**Binary Relations**

A binary relation $R$ on a set $S$ is a subset of $R \times S$.

Examples:
The edge set of a graph $G = (V, E)$ with vertices $V$ is a binary relation on $V$. Write in infix form $v_1 Ev_2$ to mean $(v_1, v_2)$ is an edge in $G$.
The relation $Factor(N, M)$ on $N$ defined by $Factor(x, y)$ iff $x$ is a factor of $y$.
The relation $Sibling(P, Q)$, with the informal meaning “$P$ is a sibling of $Q$”, is a binary relation on the set of all people.

**Transitive Closure**

Given a binary relation $R$, the transitive closure $R^+$ of $R$ is the smallest transitive binary relation $S$ such that $R \subseteq S$.
The reflexive transitive closure $R^*$ of $R$ is $R^+ \cup Id$ where $Id$ is the identity relation, i.e., $I(x, x)$ for all $x$.
If $R$ is transitive, $R^+ = R$.

**Common Binary Relation Types**

A binary relation $R$ is reflexive if for all $x$, $R(x, x)$; symmetric if for all $x, y$, $R(x, y)$ implies $R(y, x)$; and transitive if for all $x, y, z$, $R(x, y)$ and $R(y, z)$ implies $R(x, z)$.

Examples:
$E$ is symmetric for undirected graphs, irreflexive if at least one vertex does not have a self-loop, and intransitive.
$Factor$ is reflexive and transitive but asymmetric.
$Sibling$ is symmetric and transitive but not reflexive.

**Characterization of Transitive Closure**

Let $R^k$ be defined recursively as follows:
$R^1 = R$.
$R^{k+1} = R \circ R^k$, where $\circ$ is relational composition.

Proposition:
$R^+ = \bigcup_{k=1}^{\infty} R^k$.

Outline proof. Easily verified that RHS is transitive and contains $R$; since LHS is the intersection of all transitive relations that contain $R$, it is contained in RHS. Then show that any proper subset of RHS cannot be transitive.
**Transitive Closure Examples**

A binary relation $R$ is an equivalence (relation) if it is reflexive, symmetric and transitive. **Fact:** An equivalence relation induces a partition on its set.

Examples:
(i) The relation $\text{path}(u,v)$ in an undirected self-looped graph $G$ with meaning “there is a path from vertex $u$ to vertex $v$”.
(ii) The relation $x \equiv y \pmod{k}$ on numbers, with meaning $x$ and $y$ have the same remainder on division by $k$.

A partition of a set $S$ is a disjoint collection of subsets of $S$ whose union is $S$. Informally, a partition of $S$ divides it up into disjoint pieces.

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**What is wrong with this “proof”?**

(False) Assertion: If a binary relation $R$ is symmetric and transitive then it is reflexive.

(Wrong) Proof:
Write in infix form. Suppose $xRy$. Then by symmetry $yRx$. Hence by transitivity $xRx$. But $x$ was arbitrary, hence $R$ is reflexive.

Diagnosis: Consider the relation $R$ on the set of all people where $xRy$ means “$x$ can see $y$”. Think of a blind person $x_0$.

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**Equivalence Relations**

$R$ is an equivalence on $S$. Denote by $[u]$ the set of all elements of $S$ that are $R$-related to $u$, i.e. $[u] = \{v | u \in S \land uRv\}$ (Notice that since $R$ is transitive, if there is a sequence $u = x_1, x_2, \ldots, x_n = v$ such that $x_1Rx_2, x_2Rx_3, \ldots, x_{n-1}Rx_n$, then $x_1Rx_n$, i.e., $uRv$.)

It is not hard to show that if $x$ and $y$ are not $R$-related, then $[x] \cap [y] = \emptyset$.

Examples:
In (i) the partitions are the separate connected components of $G$. In (ii) the partitions are the residue classes mod $k$. 

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