1. **Strange Loops**: The following system, based on a system called MiU, is perhaps famously mentioned in Douglas Hofstadter’s book, *Gödel, Escher, Bach*.

\[
\begin{align*}
\text{MI} & \xrightarrow{1} \text{MIU} \\
\text{MIU} & \xrightarrow{2} \text{MIU} \\
\text{MIU} & \xrightarrow{3} \text{MIU} \\
\text{MIU} & \xrightarrow{4} \text{MIU} \\
\text{MIU} & \xrightarrow{5} \text{MIU}
\end{align*}
\]

(a) Is MiU derivable? If so, show the derivation tree. If not, explain why not.

**Solution:**

\[
\begin{align*}
\text{MI} & \xrightarrow{1} \text{MIU} \\
\text{MIU} & \xrightarrow{2} \text{MIU} \\
\text{MIU} & \xrightarrow{3} \text{MIU} \\
\text{MIU} & \xrightarrow{4} \text{MIU} \\
\text{MIU} & \xrightarrow{5} \text{MIU}
\end{align*}
\]

(b) Is \(\frac{xIU}{xI}\) admissible? Is it derivable? Justify your answer.

**Solution:** It is not derivable, but it is admissible. It is not derivable because there is no way to construct a tree that looks like this:

\[
\begin{align*}
\text{xIU} & \xrightarrow{1} \text{MIU} \\
\text{MIU} & \xrightarrow{2} \text{MIU}
\end{align*}
\]

It is, however, admissible because it does not change the language MiU. There is no string \(x\) that could be judged \(x \text{ MIU}\) with this rule that could not be so judged without it. We could show this by proving the rule using rule induction.

(c) **Tricky**: Perhaps famously, \(\text{MU MiU}\) is not admissible. Prove this using rule induction. *Hint:* Try proving something related to the number of Is in the string.

**Solution:** We will prove that the number of Is in any string in MiU is not divisible by three. Seeing as \(\text{MU}\) has zero Is (a multiple of three), if we prove the above, we prove that \(\text{MU}\) is not admissible.

**Base Case (From rule 1).** We see that the string \(\text{MI}\) has only one I, which is not a multiple of three, hence we have shown our goal.

**Inductive case (From rule 2).** Given that the number of Is in \(xI\) is not divisible by three (our inductive hypothesis), we can easily see that the number of Is in \(xIU\) is identical and therefore is similarly not divisible by three.

**Inductive case (From rule 3).** Let \(n\) be the number of Is in \(\text{Mx}\). Our inductive hypothesis is that \(3 \nmid n\). The number of Is in \(\text{Mxx}\), clearly \(2n\), is similarly indivisible, i.e. \(3 \nmid n \implies 3 \nmid 2n\).
Inductive case (From rule 4). Let $n$ be the number of $I$s in $xIIIy$. Our inductive hypothesis is that $3 \nmid n$. The number of $I$s in $xIly$, clearly $n - 3$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid (n - 3)$

Inductive case (From rule 5). Given that the number of $I$s in $xIIIy$ is the same as the number of $I$s in $xy$, our inductive hypothesis trivially proves our goal.

Thus, by induction, no string in $MiU$ has a number of $I$s divisible by three. Therefore, $MU MiU$ is not admissible.

(d) Here is another language, which we’ll call $M_i$:

\[
\begin{align*}
\text{Mi} & \quad A \quad \text{Mi} \\
\text{Mx Mi} & \quad B \\
x IIII I y Mi & \quad C
\end{align*}
\]

\[
\begin{align*}
\text{i. Prove using rule induction that all strings in Mi could be expressed as follows, for some } k \text{ and some } i, \text{ where } 2^k - 6i > 0 \text{ (where } C^n \text{ is the character } C \text{ repeated } n \text{ times):}
\end{align*}
\]

\[
\text{Mi}^{2^k - 6i}
\]

\[
\text{Solution:}
\]

Base case (From rule A). $\text{Mi} = \text{Mi}^{2^k - 6i}$ when $2^k - 6i = 1$, i.e when $k = 0$ and $i = 0$.

Inductive case (From rule B) Given that $\text{Mx} = \text{Mi}^{2^k - 6b}$ (our inductive hypothesis), we must show that $\text{Mx Mi} = \text{Mi}^{2^k - 6i}$ for some $k$ and some $i$. As $x = I^{2^k - 6b}$ (from I.H), it is easy to see that $xx = I^{2(2^k - 6b)} = I^{2^{k+1} - 6(2b)} = I^{2^k - 6i}$ for $k = a + 1$ and $i = 2b$.

Inductive case (from rule C) Given that $x IIII I y = \text{Mi}^{2^k - 6b}$ (our inductive hypothesis). We must show that $xy = \text{Mi}^{2^k - 6i}$ for some $k$ and $i$. It should be clear to see that this rule simply subtracts six $I$ characters, and therefore $xy = \text{Mi}^{2^k - 6(b+1)}$, hence $k = a$ and $i = b + 1$.

Thus, all strings in $Mi$ can be expressed as $\text{Mi}^{2^k - 6i}$ where $2^k - 6i > 0$.

\[
\text{ii. We will now prove the opposite claim that, for all } k \text{ and } i, \text{ assuming } 2^k - 6i > 0:}
\]

$\text{Mi}^{2^k - 6i}$

To prove this we will need a few lemmas which we will prove separately.

\text{\textbf{\alpha}}) Prove, using induction on the natural number $k$ (i.e when $k = 0$ and when $k = k' + 1$), that $\text{Mi}^{2^k}$.

\text{\textbf{Solution:}}

Base case (when $k = 0$). We have to show $\text{Mi} M_i$, which is true by rule A.

Inductive case (when $k = k' + 1$) We have to show $\text{Mi}^{2^{k' + 1}}$, with the inductive hypothesis that $\text{Mi}^{2^k}$. Equivalently, we have to show $\text{Mi}^{2^k'}$, as follows:

\[
\begin{align*}
\text{Mi}^{2^k} & \text{ I'H} \\
\text{Mi}^{2^k} & \text{ Mi}^{2^k'}
\end{align*}
\]

Therefore, by induction on the natural number $k$, we have shown $\forall k, \text{Mi}^{2^k}$.

\text{\textbf{\beta}) Prove, using induction on the natural number $i$, that $\text{Mi}^{2^k}$ implies $\text{Mi}^{2^k - 6i}$, assuming $k - 6i > 0$.}

\text{\textbf{Solution:}}

Base case (when $i = 0$). We must show that $\text{Mi}^{2^k}$ implies $\text{Mi}^{2^k - 60}$, which is obviously a tautology.
Inductive case (when $i = i' + 1$) We must show that $M I^k M$ implies $M I^{k-6(i' + 1)} M$, given the inductive hypothesis that $M I^{k-6i'} M$. Note that our I.H can be restated as $M I^{k-6(i' + 1)} M$ due to our assumption that $k - 6(i' + 1) > 0$. With this, we can prove our goal as shown:

\[
\begin{array}{c}
\text{M I}^{k-6(i' + 1)} M \quad \text{I.H} \\
\hline
\text{M I}^{k-6(i' + 1)} M \\
\end{array}
\]

Therefore, our goal is shown by induction.

Hence, as we know $M I^{2k} M$ for all $k$ from lemma $\alpha$, we can conclude from lemma $\beta$ that $M I^{2k-6i} M$ for all $k$ and all $i$ where $2^k - 6i > 0$ by modus ponens.

These two parts prove that the language $M$ is exactly characterised by the formulation $M I^{2k-6i}$ where $2^k - 6i > 0$. A very useful result!

iii. Hence prove or disprove that the following rule is admissible in $M$:

$$\frac{M x x M}{M x M}$$

Solution: We know from part i that $M x x M \implies x^2 = I^{2^k-6i}$ for some $k$ and some $i$ where $2^k - 6i > 0$.

This rule is not admissible as it adds strings to the language. As $2^4 - 6 = 10$, we know $M I^{10}$ is in the language. This rule would make $M I^5$ a string in the language which it is not as there is no $k$ and $i$ such that $2^k - 6i = 5$.

iv. Why is the following rule not admissible in $M$?

$$\frac{x y M}{x III III y}$$

Solution: The rule is not admissible as it adds strings to the language. This allows us to add six $I$ characters to any string in $M$ and judge it in $M$, which results in additional strings. For example, applying the rule to $M I$ (which is in $M$), gives us $M I^7$, when our existing formulation of $M$ clearly only allows for even amounts of $I$s.

v. Prove that, for all $s$, $s M \implies s M I u$. Note that using straightforward rule induction appears to necessitate $\text{LEM}_2$ above, which we know is not admissible. Try proving using the characterisation we have already developed.

Solution: We shall show that all strings in $M$, characterised by $M I^{2k-6i}$ where $2^k - 6i > 0$, are also in $M I u$. That is, we shall show that $M I^{2k-6i} M I u$.

To start, we shall prove inductively on $k$ that $M I^{2^k} M I u$ for all $k$.

Base case (Where $k = 0$). We must show $M I M I u$, which we know trivially from rule 1.

Inductive case (where $k = k' + 1$). We must show $M I^{2^{k'+1}} M I u$, given the inductive hypothesis that $M I^{2^{k'}} M I u$. Note we can restate our proof goal as $M I^{2^{k'}} I^{2^{k'+1}} M I u$

\[
\begin{array}{c}
\text{M I}^{2^{k'}} M I u \quad \text{I.H} \\
\hline
\text{M I}^{2^{k'}} M I u \\
\end{array}
\]

Thus we have shown by induction that $M I^{2k} M I u$ for all $k$.

Next we must prove that $M I^k M I u$ implies $M I^{k-6i} M I u$ for all $i$, assuming $k - 6i > 0$. 

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Base case (where \( i = 0 \)), we must show that \( M I^k Miu \) implies \( M I^{k-0} Miu \), which is trivially a tautology.

Inductive case (where \( i = i' + 1 \)) we must show that \( M I^{k-6(i'+1)} Miu \) given the inductive hypothesis \( M I^{k-6i'} Miu \). As we know \( k - 6(i' + 1) > 0 \), we can restate our inductive hypothesis as \( M I^{k - 6(i' + 1)} Miu \), and easily prove our goal:

\[
\begin{array}{c}
\text{I.H} \\
M I^{k - 6(i' + 1)} \quad \text{Miu} \\
\hline
M I^{k - 6i'} \quad \text{Miu}
\end{array}
\]

Thus, by modus ponens we can see that \( M I^{2^k - 6i} Miu \) for all \( k \) and \( i \) where \( 2^k - 6i > 0 \). As this is the exact characterisation of \( Miu \), we have proven that \( s Miu \Rightarrow s Miu \) for all \( s \).

2. Counting Sticks: The following language (also presented in a similar form by Douglas Hofstadter, but the original invention is not his) is called the \( \Phi \Psi \) system. Unlike the \( Miu \) language discussed above, this language is not comprised of a single judgement, but of a ternary relation, written \( x \Phi y \Psi z \), where \( x, y \), and \( z \) are strings of hyphens (i.e. ‘-‘), which may be empty (\( \epsilon \)). The system is defined as follows:

\[
\epsilon \Phi \Psi B \\
-x \Phi y \Psi -z I
\]

(a) Prove that \(-\Phi \Psi -----\).

**Solution:**

\[
\begin{array}{c}
\epsilon \Phi \Psi B \\
-x \Phi y \Psi -z I
\end{array}
\]

(b) Is the following rule admissible? Is it derivable? Explain your answer.

\[
-x \Phi y \Psi -z I'
\]

**Solution:** It is not derivable (as it cannot be shown with a proof tree), but it is admissible. We know this because the language definition for \( \Phi \Psi \) is unambiguous, so the only way for \( -x \Phi y \Psi -z \) to hold is if this was established by rule \( I \). Therefore, we can deduce that \( x \Phi y \Psi z \), as this is the premise of rule \( I \). We can often “flip” or invert rules in this way, but only if the language definition is unambiguous.

(c) Show that \( x \Phi \epsilon \Psi x \), for all hyphen strings \( x \), by doing induction on the length of the hyphen string (where \( x = \epsilon \) and \( x = -x' \)).

**Solution:**

Base case (where \( x = \epsilon \)). We must show that \( \epsilon \Phi \epsilon \Psi \epsilon \), which is trivially true by rule \( B \).

Inductive case (where \( x = -x' \)) We have the inductive hypothesis \( x' \Phi \epsilon \Psi x' \), and must show that \( -x' \Phi \epsilon \Psi -x' \). Our goal trivially reduces to our induction hypothesis by rule \( I \).

Therefore we have shown \( x \Phi \epsilon \Psi x \) for all \( x \) by induction on \( x \).

(d) Show that if \( -x \Phi y \Psi z \) then \( x \Phi -y \Psi z \), for all hyphen strings \( x, y \), and \( z \), by doing a rule induction on the premise.

**Solution:**

Base case. (From rules \( B \) and \( I \), where \( -\Phi y \Psi -y \)). We must show that \( \epsilon \Phi -y \Psi -y \), which is trivially true by rule \( B \).
Inductive case. (From rule I, where $-x' \Phi y \Psi -z'$ ($\ast$)). We have the inductive hypothesis:

$$
\frac{-x' \Phi y \Psi z'}{x' \Phi -y \Psi z'} I.H
$$

We must show that $-x' \Phi -y \Psi -z'$.

$$
\frac{-x' \Phi y \Psi z'}{x' \Phi -y \Psi z'} I'.H
\frac{x' \Phi -y \Psi z'}{-x' \Phi -y \Psi -z'} I
$$

Thus we have shown by induction that if $-x \Phi y \Psi z$ then $x \Phi -y \Psi z$, for all hyphen strings $x$, $y$ and $z$.

(e) Show that $x \Phi y \Psi z$ implies $y \Phi x \Psi z$.

**Solution:** We show this by rule induction on the premise with the rules of $\Phi \Psi$.

**Base case.** (From rule B, where $\epsilon \Phi y \Psi y$). We must show that $y \Phi \epsilon y$. We proved this, most fortunately, above in part (c).

**Inductive case.** (From rule I, where $-x' \Phi y \Psi -z'$ ($\ast$)). We have the inductive hypothesis:

$$
\frac{x' \Phi y \Psi z'}{y \Phi x' \Psi z'} I.H
$$

We must show that $y \Phi -x' \Psi -z'$.

$$
\frac{-x' \Phi y \Psi -z'}{x' \Phi y \Psi z'} I'.H
\frac{x' \Phi -y \Psi z'}{y \Phi -x' \Psi -z'} (d)
$$

Thus we have shown by induction that if $x \Phi y \Psi z$ then $y \Phi x \Psi z$.

(f) Have you figured out what the $\Phi \Psi$ system actually is? Prove that if $-x \Phi -y \Psi -z$, then $z = -x+y$ (where $-z$ is a hyphen string of length $x$).

**Solution:** We proceed by rule induction on the premise.

**Base case.** (From rule B, where $-0 \Phi -y \Psi -y$), we must show that $-0 \Phi -y \Psi -0+y$. As $0 + y = y$, this trivially reduces to the premise.

**Inductive case** (From rule I, where $-x' \Phi -y \Psi -z' +1$ ($\ast$)), we have the inductive hypothesis that $-x' \Phi -y \Psi -z' \implies z' = x' + y$. We must show that $z' + 1 = (x' + 1) + y$, or, equivalently, that $z' = x' + y$:

$$
\frac{-x' \Phi -y \Psi -z'}{-x' \Phi -y \Psi -z'} I.
\frac{-x' \Phi -y \Psi -z'}{z' = x' + y} I.H
$$

Thus we have shown by rule induction that the $\Phi \Psi$ system is in fact unary addition.

3. Ambiguity and Simultaneity: Here is a simple grammar for a functional programming language $^1$:

$$
\frac{x \in \mathbb{N}}{x \text{ Expr} \text{ VAR.} \quad e_1 \text{ Expr} \quad e_2 \text{ Expr} \text{ Appl.} \quad e \text{ Expr} \quad \lambda e \text{ Expr} \text{ ABST.} \quad e \text{ Expr} \text{ PAREN.}}
$$

$^1$if you’re interested, it’s called lambda calculus, with de Bruijn indices syntax, not that it’s relevant to the question!
(a) Is this grammar ambiguous? If not, explain why not. If so, give an example of an expression that has multiple parse trees.

**Solution:** Yes, the expression 1 2 3 could be parsed two different ways, i.e:

- \[
\begin{array}{c}
1 \text{ Expr} \\
\text{Var.} \\
2 \text{ Expr} \\
\text{Var.} \\
1 2 \text{ Expr} \\
\text{Appl.} \\
3 \text{ Expr} \\
\text{Var.} \\
1 2 3 \text{ Expr} \\
\text{Appl.}
\end{array}
\]

Or:

- \[
\begin{array}{c}
1 \text{ Expr} \\
\text{Var.} \\
2 \text{ Expr} \\
\text{Var.} \\
2 3 \text{ Expr} \\
\text{Appl.} \\
1 \text{ Expr} \\
\text{Var.} \\
1 2 3 \text{ Expr} \\
\text{Appl.}
\end{array}
\]

(b) Develop a new (unambiguous) grammar that encodes the left associativity of application, that is 1 2 3 4 should be parsed as ((1 2) 3) 4 (modulo parentheses). Furthermore, lambda expressions should extend as far as possible, i.e \(\lambda 1 \ 2\) is equivalent to \(\lambda(1 \ 2)\) not \(\lambda 1 \ 2\).

**Solution:**

\[
\begin{array}{c}
x \in \mathbb{N} \quad \text{AVAR.} \\
x \text{ AExpr} \\
\quad e_1 \text{ PExpr} \\
\quad e_2 \text{ AExpr} \\
\quad e_1, e_2 \text{ PExpr} \\
\quad \text{AAppl.} \\
\quad \text{e LExpr} \\
\quad \text{AABS.} \\
\quad (e) \text{ AExpr} \\
\quad \text{APAREN.y} \\
\quad e \text{ PExpr} \\
\quad e \text{ LExpr} \\
\quad \text{SHUNT1} \\
\quad e \text{ AExpr} \\
\quad e \text{ PExpr} \\
\quad \text{SHUNT2}
\end{array}
\]

(c) Tricky Prove that all expressions in your grammar are representable in Expr, that is, that your grammar describes only strings that are in Expr.

**Solution:** We shall prove the following simultaneously:

- \(x \text{ LExpr} \Rightarrow x \text{ Expr}\)
- \(x \text{ PExpr} \Rightarrow x \text{ Expr}\)
- \(x \text{ AExpr} \Rightarrow x \text{ Expr}\)

**Proof.** Base case (From rule AVAR., where \(x \text{ AExpr} \) for some \(x \in \mathbb{N}\)). We must show \(x \text{ Expr}\), trivial by rule VAR.

*Inductive case.* (From rule AAppl., where \(e_1, e_2 \text{ PExpr}\). By inversion on rule AAppl., we deduce \(e_1 \text{ PExpr} \) (⋆), and \(e_2 \text{ AExpr} \) (⋆⋆). We have the inductive hypotheses \(e_1 \text{ AExpr} \lor e_1 \text{ PExpr} \lor e_1 \text{ LExpr} \Rightarrow e_1 \text{ Expr} \) (I.H1) and \(e_2 \text{ AExpr} \lor e_2 \text{ PExpr} \lor e_2 \text{ LExpr} \Rightarrow e_2 \text{ Expr} \) (I.H2). By lifting the three disjunctions out of the implications, we get six implications, essentially saying that if \(e_1\) or \(e_2\) is either \(\text{PExpr}\), \(\text{LExpr}\), or \(\text{AExpr}\), then it is \(\text{Expr}\).

We must show that \(e_1, e_2 \text{ Expr}\).

\[
\begin{array}{c}
e_1 \text{ LExpr} \ \ (\star) \\
\downarrow \quad \downarrow \\
e_1 \text{ Expr} \quad \text{IH1} \\
\quad e_1, e_2 \text{ Expr} \\
\end{array}
\]

*Inductive case.* (From rule AABS., where \(\lambda x \text{ LExpr}\). We can deduce by inversion of rule AABS. that \(x \text{ LExpr}\). Applying one of our inductive hypotheses \(x \text{ LExpr} \Rightarrow x \text{ Expr}\) to this, we can deduce that \(x \text{ Expr}\), and then we can apply forwards the rule Abs. to prove our goal: \(\lambda x \text{ Expr}\).

*Inductive case.* (From rule APAREN., where \((x) \text{ AExpr}\). We can deduce by inversion that \(x \text{ LExpr}\). Using one of the I.H, we get \(x \text{ Expr}\), then by rule Paren. we show our goal \((x) \text{ Expr}\).

The inductive case for the rules SHUNT1 and SHUNT2 are trivial as they do not alter the expression.
Thus, by induction, \( s \text{LE}xpr \lor s \text{PE}xpr \lor s \text{AE}xpr \implies s \text{Expr} \). We can state this more succinctly thanks to the SHUNT rules as \( s \text{LE}xpr \implies s \text{Expr} \).