COMP4161: Advanced Topics in Software Verification

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S2/2016

Content

➜ Intro & motivation, getting started [1]

➜ Foundations & Principles

- Lambda Calculus, natural deduction [1,2]
- Higher Order Logic [3[a]]
- Term rewriting [4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [5]
- Datatypes, recursion, induction [6, 7]
- Hoare logic, proofs about programs, C verification [8[b],9]
- (mid-semester break)
- Writing Automated Proof Methods [10]
- Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

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is **applicable** to term $t[s]$ with σ if

- $\sigma\ l = s$ and
- $\sigma\ P_1, \ldots, \sigma\ P_n$ are provable by rewriting.

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simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$\begin{array}{rcl} \neg A & \mapsto & A = False \\ A \longrightarrow B & \mapsto & A \Longrightarrow B \\ A \wedge B & \mapsto & A,\ B \\ \forall x.\ A\ x & \mapsto & A\ ?x \\ A & \mapsto & A = True \end{array}$$

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Demo

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Similar for any data type t: **t.split**

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if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
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- ➜ use locally with e.g. **apply** (simp cong: <rule>)

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For types nat, int etc:

- lemmas **add_ac** sort any sum $(+)$
- lemmas **mult_ac** sort any product $(*)$

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \leadsto \cdots \leadsto a + (b + c)$

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If these 3 rules are present for an AC operator
Isabelle will order terms correctly

Demo

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$$(1)+(3) \quad \{x \mapsto g\ z\} \quad a \xleftarrow{(1)} f\ (g\ z) \xrightarrow{(3)} b$$

$$(3)+(2) \quad \{z \mapsto y\} \quad b \xleftarrow{(3)} f\ (g\ y) \xrightarrow{(2)} f\ b$$

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This is the main idea of the Knuth-Bendix completion algorithm.

Demo: Waldmeister

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Application: functional programming languages

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