Advanced Algorithms
COMP4121

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

Order Statistics
Problem: Given $n$ elements, select the $i^{th}$ smallest element;
Let us get to work: Order Statistics algorithms

- **Problem:** Given $n$ elements, select the $i^{th}$ smallest element;
  - for $i = 1$ we get the **minimum**;
Let us get to work: Order Statistics algorithms

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- for $i = 1$ we get the **minimum**;
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Problem: Given \( n \) elements, select the \( i^{th} \) smallest element:
- for \( i = 1 \) we get the \textbf{minimum};
- for \( i = n \) we get the \textbf{maximum};
- for \( i = \left\lfloor \frac{n+1}{2} \right\rfloor \) we get the \textbf{median}.
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We can find both the minimum and the maximum in $O(n)$ many steps (linear time).
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- Clearly, we can do it in time $n \log n$, just MergeSort the array and find the middle element(s) of the sorted array.

- Can we do it faster???
We will show that this can be done in linear time, by both a deterministic and by a randomised algorithm.
Fast algorithms for finding the median

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- Why bother with a randomised algorithm if it can be done in linear time with a deterministic algorithm?

- Because in practice the randomised algorithm runs much faster, having much smaller constant $c$ in the bound for the run time $T(n) \leq c \cdot n$. 

Fast algorithms for finding the median

- We will show that this can be done in linear time, by both a deterministic and by a randomised algorithm.
- Why bother with a randomised algorithm if it can be done in linear time with a deterministic algorithm?
- Because in practice the randomised algorithm runs much faster, having much smaller constant $c$ in the bound for the run time $T(n) \leq c \cdot n$.
- It turns out that it is easier to solve (both deterministically and with randomisation) the more general problem of finding the $i^{th}$ smallest element for an arbitrary $i$ than to find just the median.
Fast randomised algorithm for finding the median

- **Problem**: Given $n$ elements, select the $i^{th}$ smallest element.

**Idea**: Divide-and-conquer; by doing “one half” of the randomised QuickSort, operating on one side of the partition only.

**Rand-Select**($A, p, r, i$): *choose the $i^{th}$ smallest elt of $A[p..r]$*

1. if $p = r$ & $i = 1$ then return $A[p]$;

2. choose a random pivot $pv$ from $A[p..r]$;

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   (fill the details how we do this in place, just as it is done in an implementation of the QuickSort)

4. $k \leftarrow q-p+1$ (is the number of elements $\leq pv = A[q]$);

5. if $k = i$ then return $A[q]$;

6. if $i < k$ then return Rand-Select($A, p, q-1, i$);

7. else return Rand-Select($A, q+1, r, i-k$).
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```plaintext
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Analysis of **RAND-SELECT** \((A, p, r, i)\)

Clearly, the worst case run time is \(\Theta(n^2)\). This happens, for example, in a very unlikely event that you always pick either the smallest or the largest element of the array. In such a case during each call of **RAND-SELECT** the size of the array drops only by 1. Due to reshuffling of elements around the pivot, each iteration of **RAND-SELECT** costs the length of the array, and you get 

\[
T(n) = c(n + (n-1) + (n-2) + \ldots + 1) = \Theta(n^2)
\]

This is very unlikely to happen; in fact, as we will now see, most of the time the partitions will be reasonably well balanced.
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- Let us first assume that all the elements in the array are distinct.
- Let us call a partition a balanced partition if the ratio between the number of elements in the smaller piece and the number of elements in the larger piece is not worse than 1 to 9 (9 is kind of arbitrary here, any small number $> 2$ would do).

What is the probability that we get a balanced partition after choosing the pivot? Clearly, this happens if we chose an element which is neither among the smallest $\frac{1}{10}$ nor among the largest $\frac{1}{10}$ of all elements. Thus, the probability to end up with a balanced partition is $1 - \frac{2}{10} = \frac{8}{10}$.

Let us find the expected number of partitions between two consecutive balanced partitions.
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Thus, the probability to end up with a balanced partition is $1 - 2/10 = 8/10$. Let us find the expected number of partitions between two consecutive balanced partitions.
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- The probability to get another balanced partition immediately after a balanced partition is $\frac{8}{10}$;
- The probability to need two partitions to get another balanced partition is $\frac{2}{10} \frac{8}{10}$;
- In general, the probability that you will need $k$ partitions to end up with another balanced partition is $\left(\frac{2}{10}\right)^{k-1} \cdot \frac{8}{10}$.
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- The probability to get another balanced partition immediately after a balanced partition is $\frac{8}{10}$;
- The probability to need two partitions to get another balanced partition is $\frac{2}{10} \cdot \frac{8}{10}$;
- In general, the probability that you will need $k$ partitions to end up with another balanced partition is $\left(\frac{2}{10}\right)^{k-1} \cdot \frac{8}{10}$.
- Thus, the expected number of partitions between two balanced partitions is

$$E = 1 \cdot \frac{8}{10} + 2 \cdot \frac{2}{10} \cdot \frac{8}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 \cdot \frac{8}{10} + \ldots$$

$$= \frac{8}{10} \cdot \sum_{k=0}^{\infty} (k+1) \left(\frac{2}{10}\right)^k = \frac{8}{10} S$$

where

$$S = 1 + 2 \cdot \frac{2}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 + 4 \cdot \left(\frac{2}{10}\right)^3 + 5 \cdot \left(\frac{2}{10}\right)^4 + \ldots$$
Evaluating $S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k$

- How do we evaluate such a sum $S$?
Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

- How do we evaluate such a sum \( S \)?
- **Trick # 1:**

\[
S = 1 + 2 \cdot \frac{2}{10} + 3 \cdot \left( \frac{2}{10} \right)^2 + 4 \cdot \left( \frac{2}{10} \right)^3 + 5 \cdot \left( \frac{2}{10} \right)^4 + \ldots
\]

\[
= 1 + \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots
\]

\[
+ \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots
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\]
Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

- Summing each row separately we obtain

\[
1 + \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots = \frac{1}{1 - \frac{2}{10}} = \frac{10}{8}
\]

\[
+ \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots = \frac{2}{10} \frac{10}{8}
\]

\[
+ \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots = \left( \frac{2}{10} \right) \frac{2}{8}
\]

\[
+ \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots = \left( \frac{2}{10} \right) \frac{3}{8}
\]

\[
\ldots
\]
Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

Summing each row separately we obtain

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\]

\[
+ \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots = \left( \frac{2}{10} \right)^3 \frac{10}{8}
\]

\[
\ldots
\]

We can now sum the right hand side column:
Evaluating $S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k$

\[
S = \frac{10}{8} \left( 1 + \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots \right)
\]
\[
= \frac{10}{8} \frac{1}{1 - \frac{2}{10}} = \left( \frac{10}{8} \right)^2
\]
Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

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\[
= \frac{10}{8} \frac{1}{1 - \frac{2}{10}} = \left( \frac{10}{8} \right)^2
\]

Thus, we obtain

\[
E = \frac{8}{10} S = \frac{8}{10} \left( \frac{10}{8} \right)^2 = \frac{10}{8} = \frac{5}{4} < 2
\]
A useful digression (Trick #2): \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)
evaluated another way.

Note that

\[
\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}.
\]

By differentiating both sides with respect to \( q \) we get

\[
\sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1 - q)^2}.
\]

Substituting \( q = 2/10 \) we get that \( S = (10/8)^2 \).
Performance of \texttt{Rand-Select}:

- So, on average, there are only $5/4$ partitions between two balanced partitions.
Performance of **Rand-Select**:

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- So, on average, there are only $5/4$ partitions between two balanced partitions.
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  $$= \frac{5}{4}n \div \left(1 - \frac{9}{10}\right) = \frac{50}{4} \times n = 12.5n$$

- Where did we tacitly assume that all elements are distinct?
- How did we estimate the probability of choosing the pivot which results in a balanced partition?
Performance of **Rand-Select**: 

- Note that if all elements are the same, **Rand-Select** would run in quadratic time no matter which elements are chosen as pivots - they are all equal.

- **Homework:** Modify **Rand-Select** so that it runs in linear time even when there are many repetitions.
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- **Main idea:** Use a recursive call of the very same algorithm to choose a good pivot!
Algorithm Select\((n, i)\):
- Split the numbers in groups of five (the last group might contain less than 5 elements);
- Order each group by brute force in an increasing order.

Take the collection of all \(\lfloor \frac{n}{5} \rfloor\) middle elements of each group (i.e., the medians of each group of five).
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- Take the collection of all \(\lfloor \frac{n}{5} \rfloor\) middle elements of each group (i.e., the medians of each group of five).
- Apply recursively SELECT algorithm to find the median \(p\) of this collection.
Algorithm Select\((n, i)\) continued:

- partition all elements using \(p\) as a pivot;
- Let \(k\) be the number of elements in the subset of all elements smaller than the pivot \(p\).
- if \(i = k\) then return \(p\)
- else if \(i < k\) then recursively select the \(i\)th smallest element of the set of elements smaller than the pivot.
- else recursively select the \((i - k)\)th smallest element of the set of elements larger than the pivot.

Note: This algorithm is the same as Rand-Select except for the way how we chose the pivot. Instead of choosing pivot randomly we called recursively the very same algorithm to pick the pivot as the median of the middle elements of the groups of five elements.
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Deterministic Linear Time Algorithm for Order Statistics

- **Algorithm Select** \((n, i)\) continued:
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- Note that at least \(\lfloor (n/5)/2 \rfloor = \lfloor n/10 \rfloor\) group medians are smaller or equal to the pivot; and at least that many larger than the pivot.
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But this implies that at least \( \lfloor 3n/10 \rfloor \) of the total number of elements are smaller than the pivot, and that many elements larger than the pivot.
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Note that at least \( \lfloor (n/5)/2 \rfloor = \lfloor n/10 \rfloor \) group medians are smaller or equal to the pivot; and at least that many larger than the pivot. But this implies that at least \( \lfloor 3n/10 \rfloor \) of the total number of elements are smaller than the pivot, and that many elements larger than the pivot.

(the same caveat: we are assuming all elements are distinct; otherwise we have to slightly tweak the algorithm to split all elements equal to the pivot evenly between the two groups.)
What is the run time of our algorithm?

\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + Cn. \]

Let us show that
\[ T(n) < 11Cn \]
for all \( n \).

Assume that this is true for all \( k < n \) and let us prove it is true for \( n \) as well.

Note: this is a proof using the following form of induction:
\[ \phi(0) \land \left( \forall n \left( \left( \forall k < n \phi(k) \right) \rightarrow \phi(n) \right) \right) \rightarrow \left( \forall n \phi(n) \right). \]

Thus, assume
\[ T\left(\frac{n}{5}\right) < 11C \cdot \frac{n}{5} \]
and
\[ T\left(\frac{7n}{10}\right) < 11C \cdot \frac{7n}{10}; \]
then
\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + Cn < 11C \cdot \frac{n}{5} + 11C \cdot \frac{7n}{10} + Cn = 109Cn < 11C \cdot n. \]

which proves our statement that
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Deterministic Linear Time Algorithm for Order Statistics

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which proves out statement that \( T(n) < 11C \cdot n \).
Note that this algorithm is a genuine recursion (rather than just an iteration) so its execution involves lots of traffic on the stack, which makes this algorithm slow in practice; the randomised version of it, RAND-SELECT, significantly outperforms it.
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Similarly RAND-QUICKSORT in practice outperforms MERGESORT, which, unlike RAND-QUICKSORT, is guaranteed to run in time $O(n \log n)$. 