NICTA

COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Content

NICTA

- Intro & motivation, getting started with Isabelle
- **Foundations & Principles**
  - Lambda Calculus
  - Higher Order Logic, natural deduction
  - **Term rewriting**
- Proof & Specification Techniques
  - Inductively defined sets, rule induction
  - Datatypes, recursion, induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

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Last Time

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- Introducing new Types
- Equations and Term Rewriting
- Confluence and Termination of reduction systems
- Term Rewriting in Isabelle

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Exercises

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- use **typedef** to define a new type v with exactly one element.
- define a constant u of type v
- show that every element of v is equal to u
- design a set of rules that turns formulae with $\wedge, \vee, \longrightarrow, \neg$
  into disjunctive normal form
  (= disjunction of conjunctions with negation only directly on variables)
- prove those rules in Isabelle
- use **simp only** with these rules on $(\neg B \longrightarrow C) \longrightarrow A \longrightarrow B$

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Isar

A Language for Structured Proofs

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Isar

apply scripts

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

No structure.

What about..

- ➜ Elegance?
- ➜ Explaining deeper insights?
- ➜ Large developments?

Isar!

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A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** simp
  ⋮
  have $formula_n$ **by** blast
  show $formula_{n+1}$ **by** ...
qed

proves $formula_0 \Longrightarrow formula_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

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Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

method = (simp ...) | (blast ...) | (rule ...) | ...

statement = **fix** variables ($\bigwedge$)
  | **assume** proposition ($\Longrightarrow$)
  | [**from** name$^+$] (**have** | **show**) proposition proof
  | **next** (separates subgoals)

proposition = [name:] formula

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proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
next
  assume B: "B"
  from B **show** "B" **by** assumption
qed

- **proof** (<method>) applies method to the stated goal
- **proof** applies a single rule that fits
- **proof -** does nothing to the goal

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How do I know what to Assume and Show?

Look at the proof state!

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
proof (rule conjI)

- **proof** (rule conjI) changes proof state to
  1. $[\![A; B]\!] \Longrightarrow A$
  2. $[\![A; B]\!] \Longrightarrow B$
- so we need 2 shows: **show** "A" and **show** "B"
- We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

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The Three Modes of Isar

- **[prove]**:
  goal has been stated, proof needs to follow.
- **[state]**:
  proof block has openend or subgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** ...

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Have

Can be used to make intermediate steps.

Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc } x$" **by** simp
  have B: "$1 + x = \text{Suc } x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

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Demo: Isar Proofs

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Back to Term Rewriting ...

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Applying a Rewrite Rule

- $l \longrightarrow r$ **applicable** to term $t[s]$
  if there is substitution σ such that $\sigma\ l = s$
- **Result:** $t[\sigma\ r]$
- **Equationally:** $t[s] = t[\sigma\ r]$

Example:

Rule: $0 + n \longrightarrow n$

Term: $a + (0 + (b + c))$

Substitution: $\sigma = \{n \mapsto b + c\}$

Result: $a + (b + c)$

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Conditional Term Rewriting

Rewrite rules can be conditional:

$$[\![P_1 \dots P_n]\!] \Longrightarrow l = r$$

is **applicable** to term $t[s]$ with σ if

- $\sigma\ l = s$ and
- $\sigma\ P_1, \dots, \sigma\ P_n$ are provable by rewriting.

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Rewriting with Assumptions

Last time: Isabelle uses assumptions in rewriting.

Can lead to non-termination.

Example:

lemma "$f\ x = g\ x \wedge g\ x = f\ x \Longrightarrow f\ x = 2$"

simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$
\begin{array}{rcl}
\neg A & \mapsto & A = False \\
A \longrightarrow B & \mapsto & A \Longrightarrow B \\
A \wedge B & \mapsto & A,\ B \\
\forall x.\ A\ x & \mapsto & A\ ?x \\
A & \mapsto & A = True
\end{array}
$$

Example: $(p \longrightarrow q \wedge \neg r) \wedge s$

$$\mapsto$$

$$p \Longrightarrow q = True \qquad r = False \qquad s = True$$

DEMO

Case splitting with simp

P (if A then s else t)
=
$(A \longrightarrow P\ s) \wedge (\neg A \longrightarrow P\ t)$

Automatic

P (case e of $0 \Rightarrow a$ | Suc $n \Rightarrow b$)
=
$(e = 0 \longrightarrow P\ a) \wedge (\forall n.\ e =$ Suc $n \longrightarrow P\ b)$

Manually: apply (simp split: nat.split)

Similar for any data type t: **t.split**

Congruence Rules

congruence rules are about using context

Example: in $P \longrightarrow Q$ we could use P to simplify terms in Q

For $\Longrightarrow$ hardwired (assumptions used in rewriting)

For other operators expressed with conditional rewriting.

Example: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \longrightarrow Q) = (P' \longrightarrow Q')$

Read: to simplify $P \longrightarrow Q$

- ➜ first simplify P to P'
- ➜ then simplify Q to Q' using P' as assumption
- ➜ the result is $P' \longrightarrow Q'$

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More Congruence

Sometimes useful, but not used automatically (slowdown):
conj_cong: $[\![P = P'; P' \Longrightarrow Q = Q']\!] \Longrightarrow (P \wedge Q) = (P' \wedge Q')$

Context for if-then-else:
if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
$(\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$

Prevent rewriting inside then-else (default):
if_weak_cong: $b = c \Longrightarrow (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } x \text{ else } y)$

- ➜ declare own congruence rules with **[cong]** attribute
- ➜ delete with **[cong del]**

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Ordered rewriting

Problem: $x + y \longrightarrow y + x$ does not terminate

Solution: use permutative rules only if term becomes lexicographically smaller.

Example: $b + a \leadsto a + b$ but not $a + b \leadsto b + a$.

For types nat, int etc:

- lemmas **add_ac** sort any sum $(+)$
- lemmas **times_ac** sort any product $(*)$

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \leadsto \cdots \leadsto a + (b + c)$

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AC Rules

Example for associative-commutative rules:

Associative: $(x \odot y) \odot z = x \odot (y \odot z)$
Commutative: $x \odot y = y \odot x$

These 2 rules alone get stuck too early (not confluent).

Example: $(z \odot x) \odot (y \odot v)$
We want: $(z \odot x) \odot (y \odot v) = v \odot (x \odot (y \odot z))$
We get: $(z \odot x) \odot (y \odot v) = v \odot (y \odot (x \odot z))$

We need: **AC rule** $x \odot (y \odot z) = y \odot (x \odot z)$

If these 3 rules are present for an AC operator
Isabelle will order terms correctly

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Demo

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Back to Confluence

Last time: confluence in general is undecidable.
But: confluence for terminating systems is decidable!
Problem: overlapping lhs of rules.

Definition:

Let $l_1 \longrightarrow r_1$ and $l_2 \longrightarrow r_2$ be two rules with disjoint variables.
They form a **critical pair** if a non-variable subterm of l_1 unifies with l_2.

Example:
Rules: (1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$
Critical pairs:

(1)+(3)	$\{x \mapsto g\ z\}$	$a \overset{(1)}{\longleftarrow} f\ g\ t \overset{(3)}{\longrightarrow} b$
(3)+(2)	$\{z \mapsto y\}$	$b \overset{(3)}{\longleftarrow} f\ g\ t \overset{(2)}{\longrightarrow} b$

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Completion

(1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$

is not confluent

But it can be made confluent by adding rules!

How: join all critical pairs

Example:

(1)+(3) $\{x \mapsto g\ z\}$ $a \overset{(1)}{\longleftarrow} f\ g\ t \overset{(3)}{\longrightarrow} b$

shows that $a = b$ (because $a \overset{*}{\longleftrightarrow} b$), so we add $a \longrightarrow b$ as a rule

This is the main idea of the Knuth-Bendix completion algorithm.

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Demo: Waldmeister

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We have learned today ...

- ➜ Isar
- ➜ Conditional term rewriting
- ➜ Congruence rules
- ➜ AC rules
- ➜ More on confluence

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