COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Content

- Intro & motivation, getting started with Isabelle
- Foundations & Principles
  - Lambda Calculus
  - Higher Order Logic, natural deduction
  - Term rewriting
- **Proof & Specification Techniques**
  - Inductively defined sets, rule induction
  - **Datatypes, recursion, induction**
  - Well founded recursion, Calculational reasoning
  - Hoare logic, proofs about programs
  - Locales, Presentation

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Datatypes

Example:

datatype 'a list = Nil | Cons 'a "'a list"

Properties:

- Constructors:

  Nil :: 'a list
  Cons :: 'a ⇒ 'a list ⇒ 'a list

- Distinctness: Nil ≠ Cons x xs
- Injectivity: (Cons x xs = Cons y ys) = (x = y ∧ xs = ys)

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The General Case

datatype $(\alpha_1, \ldots, \alpha_n)\ \tau$ = $C_1\ \tau_{1,1}\ \ldots\ \tau_{1,n_1}$
| $\ldots$
| $C_k\ \tau_{k,1}\ \ldots\ \tau_{k,n_k}$

- Constructors: $C_i :: \tau_{i,1} \Rightarrow \ldots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\ \tau$
- Distinctness: $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- Injectivity: $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity applied automatically

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How is this Type Defined?

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datatype 'a list = Nil | Cons 'a "'a list"

- ➜ internally defined using typedef
- ➜ hence: describes a set
- ➜ set = trees with constructors as nodes
- ➜ inductive definition to characterize which trees belong to datatype

More detail: Datatype_Universe.thy

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Datatype Limitations

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Must be definable as set.

- ➜ Infinitely branching ok.
- ➜ Mutually recursive ok.
- ➜ Stricly positive (right of function arrow) occurence ok.

Not ok:

datatype t = C (t ⇒ bool)
| D ((bool ⇒ t) ⇒ bool)
| E ((t ⇒ bool) ⇒ bool)

Because: Cantor's theorem (α set is larger than α)

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Case

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Every datatype introduces a **case** construct, e.g.

(case xs of [] ⇒ $\ldots$ | $y\ \#ys$ ⇒ ... y ... ys ...)

In general: one case per constructor

- ➜ Same order of cases as in datatype
- ➜ Nested patterns allowed: $x\#y\#zs$
- ➜ Binds weakly, needs $()$ in context

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Cases

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apply (case_tac t)

creates k subgoals

$$[\![t = C_i\ x_1 \ldots x_p; \ldots]\!] \Longrightarrow \ldots$$

one for each constructor C_i

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DEMO

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RECURSION

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Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

Subtract $f\ x$ on both sides.

$$\Longrightarrow$$
$$0 = 1$$

! **All functions in HOL must be total** **!**

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Primitive Recursion

primrec guarantees termination structurally

Example primrec def:

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

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The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$\begin{array}{lcl} f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\ \vdots & & \\ f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k \end{array}$$

The recursive calls in r_i must be **structurally smaller**
(of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

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How does this Work?

primrec just fancy syntax for a **recursion operator**

Example: list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

$$\begin{array}{lcl} \text{list\_rec}\ f_1\ f_2\ \text{Nil} & = & f_1 \\ \text{list\_rec}\ f_1\ f_2\ (\text{Cons}\ x\ xs) & = & f_2\ x\ xs\ (\text{list\_rec}\ f_1\ f_2\ xs) \end{array}$$

app $\equiv$ list_rec $(\lambda ys.\ ys)\ (\lambda x\ xs\ xs'.\ \lambda ys.\ \text{Cons}\ x\ (xs'\ ys))$

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"

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list_rec

Defined: automatically, first inductively (set), then by epsilon

$$\frac{}{(\text{Nil}, f_1) \in \text{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \text{list\_rel}\ f_1\ f_2}{(\text{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \text{list\_rel}\ f_1\ f_2}$$

list_rec $f_1\ f_2\ xs \equiv$ SOME $y.\ (xs, y) \in$ list_rel $f_1\ f_2$

Automatic proof that set def indeed is total function
(the equations for list_rec are lemmas!)

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PREDEFINED DATATYPES

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nat is a datatype

datatype nat $=$ 0 | Suc nat

Functions on nat definable by primrec!

primrec

$$
\begin{aligned}
f\ 0 &= \ldots \\
f\ (\text{Suc}\ n) &= \ldots\ f\ n\ \ldots
\end{aligned}
$$

Option

datatype 'a option = None | Some 'a

Important application:

'b ⇒ 'a option	∼	partial function:
None	∼	no result
Some a	∼	result a

Example:

primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
where
lookup k [] = None |
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

DEMO: PRIMREC

INDUCTION

Structural induction

$P\ xs$ holds for all lists xs if

- ➜ P Nil
- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x\#xs)$

  Induction theorem **list.induct:**
  $[\![P\ [\,];\ \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$

- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

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Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

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Example

A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
where
itrev [] $ys = ys$ |
itrev $(x\#xs)$ $ys =$ itrev xs $(x\#ys)$

lemma itrev xs [] = rev xs

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Demo: Proof Attempt

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Generalisation

Replace constants by variables

lemma itrev $xs\ ys = $ rev $xs@ys$

Quantify free variables by $\forall$
(except the induction variable)

lemma $\forall ys.$ itrev $xs\ ys = $ rev $xs@ys$

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We have seen today ...

- ➜ Datatypes
- ➜ Primitive recursion
- ➜ Case distinction
- ➜ Structural Induction

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