COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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HOL

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Content

[a] a1 due; [b] a2 due; [c] session break; [d] a3 due

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DEFINING HIGHER ORDER LOGIC

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What is Higher Order Logic?

➜ **Propositional Logic:**

- no quantifiers
- all variables have type bool

➜ **First Order Logic:**

- quantification over values, but not over functions and predicates,
- terms and formulas syntactically distinct

➜ **Higher Order Logic:**

- quantification over everything, including predicates
- consistency by types
- formula = term of type bool
- definition built on $\lambda^{\rightarrow}$ with certain default types and constants

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Defining Higher Order Logic

Default types:

bool $\qquad\qquad$ $\_ \Rightarrow \_$ $\qquad\qquad$ ind

- **bool** sometimes called o
- $\Rightarrow$ sometimes called fun

Default Constants:

$$\begin{array}{lll} \longrightarrow & :: & bool \Rightarrow bool \Rightarrow bool \\ = & :: & \alpha \Rightarrow \alpha \Rightarrow bool \\ \epsilon & :: & (\alpha \Rightarrow bool) \Rightarrow \alpha \end{array}$$

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Higher Order Abstract Syntax

Problem: Define syntax for binders like $\forall$, $\exists$, ε

One approach: $\forall :: var \Rightarrow term \Rightarrow bool$
Drawback: need to think about substitution, α conversion again.

But: Already have binder, substitution, α conversion in meta logic

$$\lambda$$

So: Use λ to encode all other binders.

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Higher Order Abstract Syntax

Example:

ALL $:: (\alpha \Rightarrow bool) \Rightarrow bool$

HOAS	**usual syntax**
ALL $(\lambda x.\ x = 2)$	$\forall x.\ x = 2$
ALL P	$\forall x.\ P\ x$

Isabelle can translate usual binder syntax into HOAS.

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Side Track: Syntax Declarations in Isabelle

- **mixfix:**
  consts drvbl $:: ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ $(''\_, \_ \vdash \_'')$
  Legal syntax now: $\Gamma, \Pi \vdash F$
- **priorities:**
  pattern can be annotated with priorities to indicate binding strength
  Example: drvbl $:: ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ $(''\_, \_ \vdash \_''\ [30, 0, 20]\ 60)$
- **infixl/infixr**: short form for left/right associative binary operators
  Example: or $:: bool \Rightarrow bool \Rightarrow bool$ (infixr $'' \lor ''$ 30)
- **binders:** declaration must be of the form
  $c :: (\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder $''B''$ $< p >$)
  $B\ x.\ P\ x$ translated into $c\ P$ (and vice versa)
  Example ALL $:: (\alpha \Rightarrow bool) \Rightarrow bool$ (binder $''\forall''$ 10)

More (including pretty printing) in Isabelle Reference Manual (7.3)

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Back to HOL

Base: $bool, \Rightarrow, ind$ $\quad =, \longrightarrow, \varepsilon$

And the rest is definitions:

$\mathsf{True} \equiv (\lambda x :: bool.\ x) = (\lambda x.\ x)$
$\mathsf{All}\ P \equiv P = (\lambda x.\ \mathsf{True})$
$\mathsf{Ex}\ P \equiv \forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q$
$\mathsf{False} \equiv \forall P.\ P$
$\neg P \equiv P \longrightarrow \mathsf{False}$
$P \wedge Q \equiv \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R$
$P \vee Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$
$\mathsf{If}\ P\ x\ y \equiv \mathsf{SOME}\ z.\ (P = \mathsf{True} \longrightarrow z = x) \wedge (P = \mathsf{False} \longrightarrow z = y)$
$\mathsf{inj}\ f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$
$\mathsf{surj}\ f \equiv \forall y.\ \exists x.\ y = f\ x$

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The Axioms of HOL

$$\frac{}{t = t}\ \mathsf{refl} \qquad \frac{s = t \quad P\ s}{P\ t}\ \mathsf{subst} \qquad \frac{\bigwedge x.\ f\ x = g\ x}{(\lambda x.\ f\ x) = (\lambda x.\ g\ x)}\ \mathsf{ext}$$

$$\frac{P \Longrightarrow Q}{P \longrightarrow Q}\ \mathsf{impI} \qquad \frac{P \longrightarrow Q \quad P}{Q}\ \mathsf{mp}$$

$$\frac{}{(P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \longrightarrow (P = Q)}\ \mathsf{iff}$$

$$\frac{}{P = \mathsf{True} \vee P = \mathsf{False}}\ \mathsf{True\_or\_False}$$

$$\frac{P\ ?x}{P\ (\mathsf{SOME}\ x.\ P\ x)}\ \mathsf{someI}$$

$$\frac{}{\exists f :: ind \Rightarrow ind.\ \mathsf{inj}\ f \wedge \neg \mathsf{surj}\ f}\ \mathsf{infty}$$

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That's it.

- ➜ 3 basic constants
- ➜ 3 basic types
- ➜ 9 axioms

With this you can define and derive all the rest.

Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y}\ \mathsf{eq\_reflection} \qquad \frac{}{(\mathsf{THE}\ x.\ x = a) = a}\ \mathsf{the\_eq\_trivial}$$

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Demo: The Definitions in Isabelle

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Deriving Proof Rules

In the following, we will

- ➜ look at the definitions in more detail
- ➜ derive the traditional proof rules from the axioms in Isabelle

Convenient for deriving rules: **named assumptions in lemmas**

lemma $[name :]$
assumes $[name_1 :]$ "$< prop >_1$"
assumes $[name_2 :]$ "$< prop >_2$"
$\vdots$
shows "$< prop >$" $< proof >$

proves: $[\![< prop >_1; < prop >_2; \ldots]\!] \Longrightarrow < prop >$

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True

consts True :: $bool$
True $\equiv (\lambda x :: bool.\ x) = (\lambda x.\ x)$

Intuition:
right hand side is always true

Proof Rules:

$$\frac{}{\text{True}}\ \text{TrueI}$$

Proof:

$$\frac{\overline{(\lambda x :: bool.\ x) = (\lambda x.\ x)}\ \text{refl}}{\text{True}}\ \text{unfold True\_def}$$

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DEMO

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Universal Quantifier

consts ALL :: $(\alpha \Rightarrow bool) \Rightarrow bool$
ALL $P \equiv P = (\lambda x.\ \text{True})$

Intuition:

- ➜ ALL P is Higher Order Abstract Syntax for $\forall x.\ P\ x$.
- ➜ P is a function that takes an x and yields a truth values.
- ➜ ALL P should be true iff P yields true for all x, i.e. if it is equivalent to the function $\lambda x.$ True.

Proof Rules:

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

Proof: Isabelle Demo

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False

consts False :: *bool*
False $\equiv$ $\forall P. P$

Intuition:
Everything can be derived from *False*.

Proof Rules:

$$\frac{\text{False}}{P}\ \text{FalseE} \qquad \frac{}{\text{True} \neq \text{False}}$$

Proof: Isabelle Demo

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Negation

consts Not :: $bool \Rightarrow bool$ $(\neg\ \_)$
$\neg P \equiv P \longrightarrow \text{False}$

Intuition:
Try $P = True$ and $P = False$ and the traditional truth table for $\longrightarrow$.

Proof Rules:

$$\frac{A \Longrightarrow False}{\neg A}\ \text{notI} \qquad \frac{\neg A \quad A}{P}\ \text{notE}$$

Proof: Isabelle Demo

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Existential Quantifier

consts EX :: $(\alpha \Rightarrow bool) \Rightarrow bool$
EX P $\equiv$ $\forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q$

Intuition:

- ➜ EX P is HOAS for $\exists x.\ P\ x$. (like $\forall$)
- ➜ Right hand side is characterization of $\exists$ with $\forall$ and $\longrightarrow$
- ➜ Note that inner $\forall$ binds wide: $(\forall x.\ P\ x \longrightarrow Q)$
- ➜ Remember lemma from last time: $(\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$

Proof Rules:

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

Proof: Isabelle Demo

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Conjunction

consts And :: $bool \Rightarrow bool \Rightarrow bool$ $(\_ \wedge \_)$
$P \wedge Q \equiv \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

Intuition:

- ➜ Mirrors proof rules for $\wedge$
- ➜ Try truth table for P, Q, and R

Proof Rules:

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

Proof: Isabelle Demo

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Disjunction

consts Or :: $bool \Rightarrow bool \Rightarrow bool$ ($\_ \vee \_$)

$P \vee Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

Intuition:

- ➜ Mirrors proof rules for $\vee$ (case distinction)
- ➜ Try truth table for P, Q, and R

Proof Rules:

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

Proof: Isabelle Demo

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If-Then-Else

consts If :: $bool \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)

If $P\ x\ y\ \equiv$ SOME $z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- ➜ for $P = \text{True}$, right hand side collapses to SOME $z.\ z = x$
- ➜ for $P = \text{False}$, right hand side collapses to SOME $z.\ z = y$

Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s}\ \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t}\ \text{ifFalse}$$

Proof: Isabelle Demo

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THAT WAS HOL

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More on Automation

Last time: safe and unsafe rule, heuristics: use safe before unsafe

This can be automated

Syntax:

[<kind>!] for safe rules (<kind> one of intro, elim, dest)

[<kind>] for unsafe rules

Application (roughly):

do safe rules first, search/backtrack on unsafe rules only

Example:

declare attribute globally	**declare** conjI [intro!] allE [elim]
remove attribute gloabllay	**declare** allE [rule del]
use locally	**apply** (blast intro: someI)
delete locally	**apply** (blast del: conjI)

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DEMO: AUTOMATION

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We have learned today ...

- ➜ Defining HOL
- ➜ Higher Order Abstract Syntax
- ➜ Deriving proof rules
- ➜ More automation

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