

COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

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HOL

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^a a1 due; ^b a2 due; ^c session break; ^d a3 due

DEFINING HIGHER ORDER LOGIC

What is Higher Order Logic?

→ Propositional Logic:

- no quantifiers
- all variables have type bool

→ First Order Logic:

- quantification over values, but not over functions and predicates,
- terms and formulas syntactically distinct

→ Higher Order Logic:

- quantification over everything, including predicates
- consistency by types
- formula = term of type bool
- definition built on λ^{\rightarrow} with certain default types and constants

Defining Higher Order Logic

Default types:

`bool` `_ \Rightarrow _` `ind`

→ `bool` sometimes called *o*

→ `\Rightarrow` sometimes called *fun*

Default Constants:

`→` `::` *bool \Rightarrow bool \Rightarrow bool*

`=` `::` *$\alpha \Rightarrow \alpha \Rightarrow bool$*

`€` `::` *($\alpha \Rightarrow bool$) $\Rightarrow \alpha$*

Higher Order Abstract Syntax

Problem: Define syntax for binders like $\forall, \exists, \varepsilon$

One approach: $\forall :: var \Rightarrow term \Rightarrow bool$

Drawback: need to think about substitution, α conversion again.

But: Already have binder, substitution, α conversion in meta logic

λ

So: Use λ to encode all other binders.

Higher Order Abstract Syntax

Example:

$$\text{ALL} :: (\alpha \Rightarrow \text{bool}) \Rightarrow \text{bool}$$

HOAS

usual syntax

$$\text{ALL } (\lambda x. x = 2)$$
$$\forall x. x = 2$$
$$\text{ALL } P$$
$$\forall x. P x$$

Isabelle can translate usual binder syntax into HOAS.

Side Track: Syntax Declarations in Isabelle

→ **mixfix:**

consts `drvbl` :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, - ⊢ -")

Legal syntax now: $\Gamma, \Pi \vdash F$

→ **priorities:**

pattern can be annotated with priorities to indicate binding strength

Example: `drvbl` :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, - ⊢ -" [30, 0, 20] 60)

→ **infixl/infixr:** short form for left/right associative binary operators

Example: `or` :: $bool \Rightarrow bool \Rightarrow bool$ (infixr " ∨ " 30)

→ **binders:** declaration must be of the form

c :: $(\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder "B" < p >)

$B x. P x$ translated into $c P$ (and vice versa)

Example `ALL` :: $(\alpha \Rightarrow bool) \Rightarrow bool$ (binder "∀" 10)

More (including pretty printing) in Isabelle Reference Manual (7.3)

Back to HOL

Base: $bool, \Rightarrow, ind \quad =, \longrightarrow, \varepsilon$

And the rest is definitions:

$\text{True} \equiv (\lambda x :: bool. x) = (\lambda x. x)$

$\text{All } P \equiv P = (\lambda x. \text{True})$

$\text{Ex } P \equiv \forall Q. (\forall x. P x \longrightarrow Q) \longrightarrow Q$

$\text{False} \equiv \forall P. P$

$\neg P \equiv P \longrightarrow \text{False}$

$P \wedge Q \equiv \forall R. (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

$P \vee Q \equiv \forall R. (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

$\text{If } P x y \equiv \text{SOME } z. (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

$\text{inj } f \equiv \forall x y. f x = f y \longrightarrow x = y$

$\text{surj } f \equiv \forall y. \exists x. y = f x$

The Axioms of HOL

$$\frac{}{t = t} \text{ refl} \quad \frac{s = t \quad P s}{P t} \text{ subst} \quad \frac{\bigwedge x. f x = g x}{(\lambda x. f x) = (\lambda x. g x)} \text{ ext}$$

$$\frac{P \implies Q}{P \longrightarrow Q} \text{ impl} \quad \frac{P \longrightarrow Q \quad P}{Q} \text{ mp}$$

$$\frac{}{(P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \longrightarrow (P = Q)} \text{ iff}$$

$$\frac{}{P = \text{True} \vee P = \text{False}} \text{ True_or_False}$$

$$\frac{P ?x}{P (\text{SOME } x. P x)} \text{ someI}$$

$$\frac{}{\exists f :: \text{ind} \implies \text{ind. inj } f \wedge \neg \text{surj } f} \text{ infty}$$

That's it.

- 3 basic constants
- 3 basic types
- 9 axioms

With this you can define and derive all the rest.

Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y} \text{ eq_reflection} \qquad \frac{}{(\text{THE } x. x = a) = a} \text{ the_eq_trivial}$$

DEMO: THE DEFINITIONS IN ISABELLE

Deriving Proof Rules

In the following, we will

- look at the definitions in more detail
- derive the traditional proof rules from the axioms in Isabelle

Convenient for deriving rules: **named assumptions in lemmas**

```
lemma [name :]  
assumes [name1 :] "< prop >1"  
assumes [name2 :] "< prop >2"  
⋮  
shows " < prop > " < proof >
```

```
proves: [ [ < prop >1; < prop >2; ... ]  $\implies$  < prop >
```

True

consts True :: *bool*

True $\equiv (\lambda x :: \text{bool}. x) = (\lambda x. x)$

Intuition:

right hand side is always true

Proof Rules:

$$\frac{}{\text{True}} \text{TrueI}$$

Proof:

$$\frac{\frac{}{(\lambda x :: \text{bool}. x) = (\lambda x. x)}{\text{True}} \text{refl}}{\text{True}} \text{unfold True_def}$$

DEMO

Universal Quantifier

consts ALL :: $(\alpha \Rightarrow bool) \Rightarrow bool$

ALL $P \equiv P = (\lambda x. \text{True})$

Intuition:

- ALL P is Higher Order Abstract Syntax for $\forall x. P x$.
- P is a function that takes an x and yields a truth value.
- ALL P should be true iff P yields true for all x , i.e.
if it is equivalent to the function $\lambda x. \text{True}$.

Proof Rules:

$$\frac{\bigwedge x. P x}{\forall x. P x} \text{ all} \qquad \frac{\forall x. P x \quad P ?x \Longrightarrow R}{R} \text{ allE}$$

Proof: Isabelle Demo

False

consts False :: *bool*

False $\equiv \forall P.P$

Intuition:

Everything can be derived from *False*.

Proof Rules:

$$\frac{\text{False}}{P} \text{ FalseE}$$
$$\overline{\text{True} \neq \text{False}}$$

Proof: Isabelle Demo

Negation

consts Not :: *bool* \Rightarrow *bool* (\neg _)

$\neg P \equiv P \longrightarrow \text{False}$

Intuition:

Try $P = \text{True}$ and $P = \text{False}$ and the traditional truth table for \longrightarrow .

Proof Rules:

$$\frac{A \Longrightarrow \text{False}}{\neg A} \text{ notI} \qquad \frac{\neg A \quad A}{P} \text{ notE}$$

Proof: Isabelle Demo

Existential Quantifier

consts EX :: $(\alpha \Rightarrow \text{bool}) \Rightarrow \text{bool}$

EX P $\equiv \forall Q. (\forall x. P x \longrightarrow Q) \longrightarrow Q$

Intuition:

- EX P is HOAS for $\exists x. P x$. (like \forall)
- Right hand side is characterization of \exists with \forall and \longrightarrow
- Note that inner \forall binds wide: $(\forall x. P x \longrightarrow Q)$
- Remember lemma from last time: $(\forall x. P x \longrightarrow Q) = ((\exists x. P x) \longrightarrow Q)$

Proof Rules:

$$\frac{P ?x}{\exists x. P x} \text{exI} \qquad \frac{\exists x. P x \quad \bigwedge x. P x \Longrightarrow R}{R} \text{exE}$$

Proof: Isabelle Demo

Conjunction

consts And :: *bool* \Rightarrow *bool* \Rightarrow *bool* ($- \wedge -$)

$P \wedge Q \equiv \forall R. (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

Intuition:

- Mirrors proof rules for \wedge
- Try truth table for P , Q , and R

Proof Rules:

$$\frac{A \quad B}{A \wedge B} \text{ conjI} \qquad \frac{A \wedge B \quad [[A; B]] \Longrightarrow C}{C} \text{ conjE}$$

Proof: Isabelle Demo

Disjunction

consts Or :: *bool* \Rightarrow *bool* \Rightarrow *bool* (*_* \vee *_*)

$P \vee Q \equiv \forall R. (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

Intuition:

- Mirrors proof rules for \vee (case distinction)
- Try truth table for P , Q , and R

Proof Rules:

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B} \quad \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C} \quad \text{disjE}$$

Proof: Isabelle Demo

If-Then-Else

consts $\text{if} :: \text{bool} \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)

$\text{if } P \ x \ y \equiv \text{SOME } z. (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- for $P = \text{True}$, right hand side collapses to $\text{SOME } z. z = x$
- for $P = \text{False}$, right hand side collapses to $\text{SOME } z. z = y$

Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s} \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t} \text{ifFalse}$$

Proof: Isabelle Demo

THAT WAS HOL

More on Automation

Last time: safe and unsafe rule, heuristics: use safe before unsafe

This can be automated

Syntax:

[<kind>!] for safe rules (<kind> one of intro, elim, dest)
 [<kind>] for unsafe rules

Application (roughly):

do safe rules first, search/backtrack on unsafe rules only

Example:

declare attribute globally	declare conjl [intro!] allE [elim]
remove attribute globally	declare allE [rule del]
use locally	apply (blast intro: some)
delete locally	apply (blast del: conjl)

DEMO: AUTOMATION

We have learned today ...

- Defining HOL
- Higher Order Abstract Syntax
- Deriving proof rules
- More automation