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^aa1 due; ^ba2 due; ^ca3 due



➔ Equations and Term Rewriting



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- → Confluence and Termination of reduction systems



- → Equations and Term Rewriting
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- → Term Rewriting in Isabelle



 \rightarrow *l* \rightarrow *r* **applicable** to term *t*[*s*]



→ I → r applicable to term t[s] if there is substitution σ such that σ I = s



- → $I \longrightarrow r$ applicable to term t[s]if there is substitution σ such that $\sigma I = s$
- → Result: $t[\sigma r]$



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Rule: $0 + n \longrightarrow n$ Term: a + (0 + (b + c))Substitution: $\sigma = \{n \mapsto b + c\}$ Result: a + (b + c)

Conditional Term Rewriting



Rewrite rules can be conditional:

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is **applicable** to term t[s] with σ if

 $\rightarrow \sigma l = s$ and

→ $\sigma P_1, \ldots, \sigma P_n$ are provable by rewriting.

Rewriting with Assumptions



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simp (simp (no_asm)) (simp (no_asm_use)) (simp (no_asm_simp))

use and simplify assumptions ignore assumptions simplify, but do not use assumptions use, but do not simplify assumptions

Preprocessing



Preprocessing (recursive) for maximal simplification power:

$$\begin{array}{cccc} \neg A & \mapsto & A = False \\ A \longrightarrow B & \mapsto & A \Longrightarrow B \\ A \land B & \mapsto & A, B \\ \forall x. \ A \ x & \mapsto & A \ ?x \\ A & \mapsto & A = True \end{array}$$

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Example:

$$(p \longrightarrow q \land \neg r) \land s$$

 \mapsto

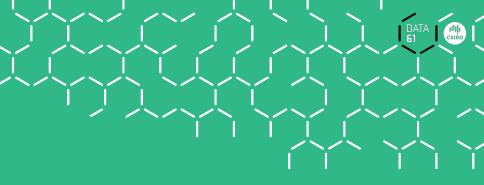
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Example: $(p \longrightarrow q \land \neg r) \land s$ \mapsto $p \Longrightarrow q = True$ $p \Longrightarrow r = False$ s = True







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$$\begin{array}{l} P \ (\text{case } e \ \text{of } 0 \ \Rightarrow \ a \mid \text{Suc } n \ \Rightarrow \ b) \\ = \\ (e = 0 \longrightarrow P \ a) \land (\forall n. \ e = \text{Suc } n \longrightarrow P \ b) \end{array}$$



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Similar for any data type t: t.split





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 - \Rightarrow the result is $P' \longrightarrow Q'$

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$$\llbracket b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v \rrbracket \Longrightarrow$$

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- → use locally with e.g. apply (simp cong: <rule>)



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For types nat, int etc:

- lemmas add_ac sort any sum (+)
- lemmas mult_ac sort any product (*)
- **Example:** apply (simp add: add_ac) yields $(b+c) + a \rightsquigarrow \cdots \rightsquigarrow a + (b+c)$



Example for associative-commutative rules: Associative: $(x \odot y) \odot z = x \odot (y \odot z)$ Commutative: $x \odot y = y \odot x$



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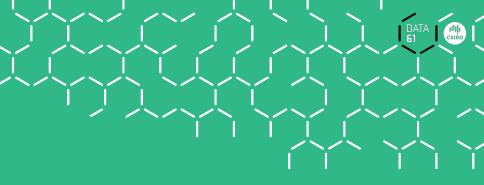
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If these 3 rules are present for an AC operator Isabelle will order terms correctly





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Definition:

Let $l_1 \longrightarrow r_1$ and $l_2 \longrightarrow r_2$ be two rules with disjoint variables.

They form a **critical pair** if a non-variable subterm of l_1 unifies with l_2 .



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$$\begin{array}{cccc} (1)+(3) & \{x \mapsto g \ z\} & a \xleftarrow{(1)} & f \ (g \ z) & \xrightarrow{(3)} b \\ (3)+(2) & \{z \mapsto y\} & b \xleftarrow{(3)} & f \ (g \ y) & \xrightarrow{(2)} f \ b \end{array}$$



(1) $f x \longrightarrow a$ (2) $g y \longrightarrow b$ (3) $f (g z) \longrightarrow b$ is not confluent

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(1) $f \times \longrightarrow a$ (2) $g \times \longrightarrow b$ (3) $f (g \times z) \longrightarrow b$ is not confluent

But it can be made confluent by adding rules!

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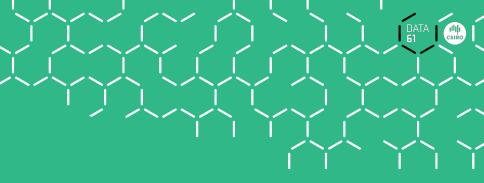
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shows that $a = b$ (because $a \stackrel{*}{\longleftrightarrow} b$), so we add $a \longrightarrow b$ as a rule

This is the main idea of the Knuth-Bendix completion algorithm.



Demo: Waldmeister



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Application: functional programming languages



➔ Conditional term rewriting



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- → AC rules



- ➔ Conditional term rewriting
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- ➔ More on confluence