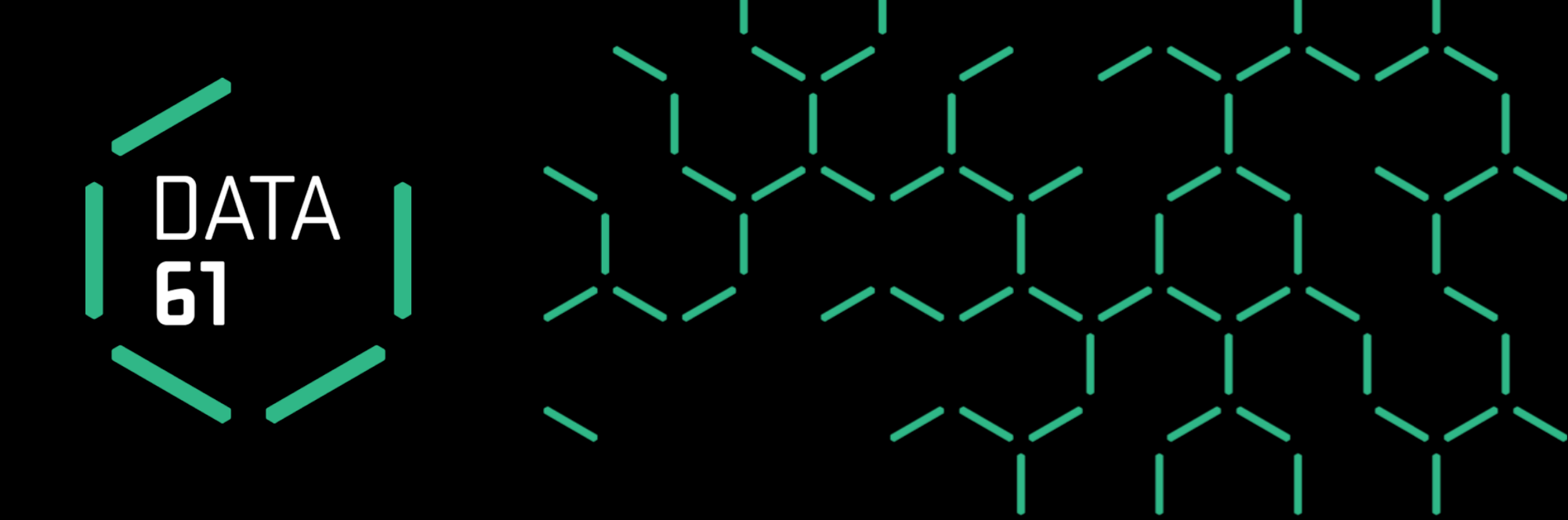
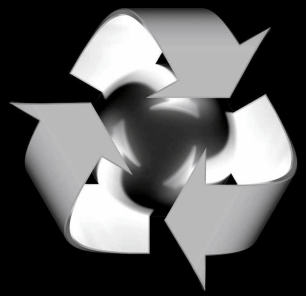


COMP4161: Advanced Topics in Software Verification



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S2/2017

data61.csiro.au

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

---

[a] a1 due; [b] a2 due; [c] a3 due

Datatypes

Example:

datatype 'a list = Nil | Cons 'a "'a list"

Properties:

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➜ Constructors:

$$\begin{array}{lll} \text{Nil} & :: & \text{'a list} \\ \text{Cons} & :: & \text{'a} \Rightarrow \text{'a list} \Rightarrow \text{'a list} \end{array}$$

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- ➜ Injectivity: $(\text{Cons x xs} = \text{Cons y ys}) = (\text{x} = \text{y} \wedge \text{xs} = \text{ys})$

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datatype answer = Yes | No | Maybe

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datatype 'a list = Nil | Cons 'a "'a list"
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Mutual Recursion:

datatype even =
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Mutual Recursion:

datatype even = EvenZero | EvenSucc odd
and odd = OddSucc even

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Nested recursion:

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➜ Recursive call is under a type constructor.

The General Case

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Distinctness and Injectivity applied automatically

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- ➜ inductive definition to characterise which trees belong to datatype

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Must be definable as set.

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Because: Cantor's theorem (α set is larger than α)

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- ➜ datatypes are automatically registered as BNF
- ➜ can register other type constructors as BNFs — not covered here**

* BNF = Bounded Natural Functors.

** *Defining (Co)datatypes and Primitively (Co)recursive Functions in Isabelle/HOL*

Case

Every datatype introduces a **case** construct, e.g.

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- ➜ Dummy and default patterns with _
- ➜ Binds weakly, needs () in context

Cases

apply (case_tac t)

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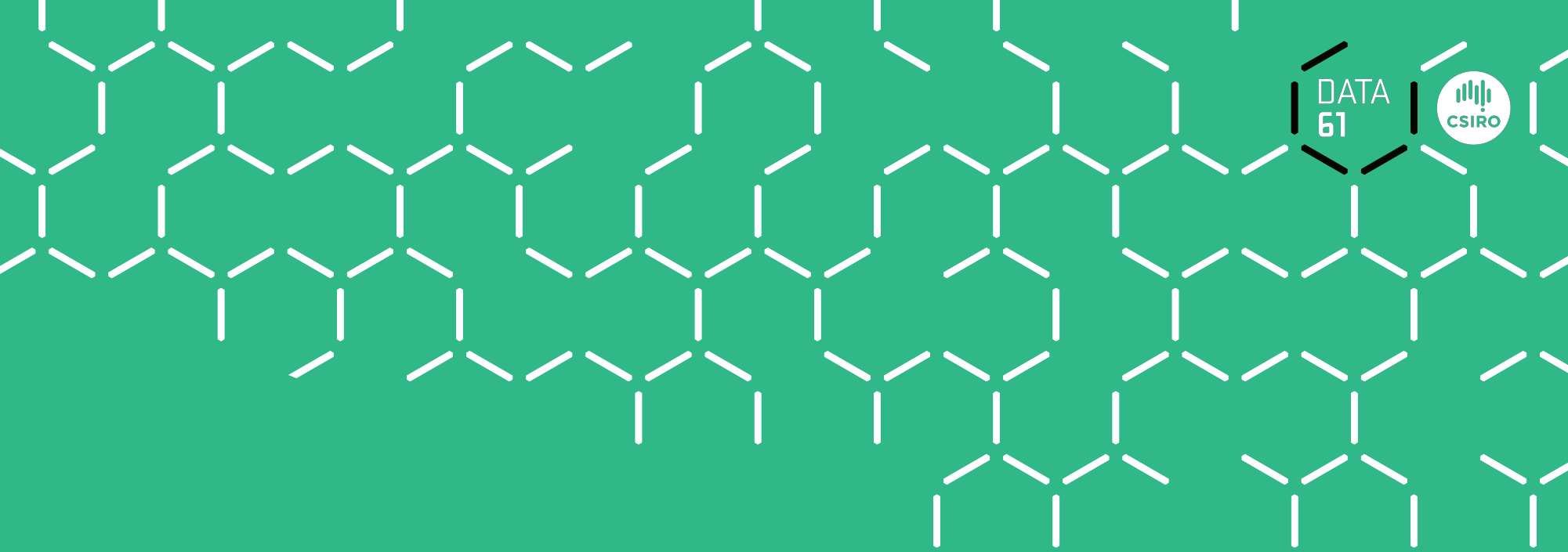
creates k subgoals

$$[\![t = C_i\ x_1 \ldots x_p; \ldots]\!] \Longrightarrow \ldots$$

one for each constructor C_i

Demo



Recursion

Why nontermination can be harmful

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! All functions in HOL must be total !

Primitive Recursion

primrec guarantees termination structurally

Example primrec def:

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Example primrec def:

```
primrec  app :: "'a list ⇒ 'a list ⇒ 'a list"
where
"app Nil ys = ys" |
"app (Cons x xs) ys = Cons x (app xs ys)"
```


The General Case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \tau \Rightarrow \tau'$ can be defined by **primitive recursion**:

$$
\begin{array}{lcl}
f\ (C_1\ y_{1,1}\ \ldots\ y_{1,n_1}) & = & r_1 \\
\vdots & & \\
f\ (C_k\ y_{k,1}\ \ldots\ y_{k,n_k}) & = & r_k
\end{array}
$$

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The recursive calls in r_i must be **structurally smaller** (of the form $f\ a_1\ \ldots\ y_{i,j}\ \ldots\ a_p$)

How does this Work?

primrec just fancy syntax for a **recursion operator**

Example:

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primrec just fancy syntax for a **recursion operator**

Example: list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

$$\begin{aligned} &\text{list\_rec } f_1\ f_2\ \text{Nil} &&= f_1 \\ &\text{list\_rec } f_1\ f_2\ (\text{Cons } x\ xs) &&= f_2\ x\ xs\ (\text{list\_rec } f_1\ f_2\ xs) \end{aligned}$$

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primrec just fancy syntax for a **recursion operator**

Example:

list_rec :: "'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"

list_rec f_1 f_2 Nil = f_1

list_rec f_1 f_2 (Cons x xs) = f_2 x xs (list_rec f_1 f_2 xs)

app ≡ list_rec ($\lambda ys.\ ys$) ($\lambda x\ xs\ xs'.\ \lambda ys.$ Cons x (xs' ys))

primrec app :: "'a list ⇒ 'a list ⇒ 'a list"
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$$\frac{}{(\text{Nil}, f_1) \in \text{list\_rel}\ f_1\ f_2} \qquad \frac{(xs, xs') \in \text{list\_rel}\ f_1\ f_2}{(\text{Cons}\ x\ xs, f_2\ x\ xs\ xs') \in \text{list\_rel}\ f_1\ f_2}$$

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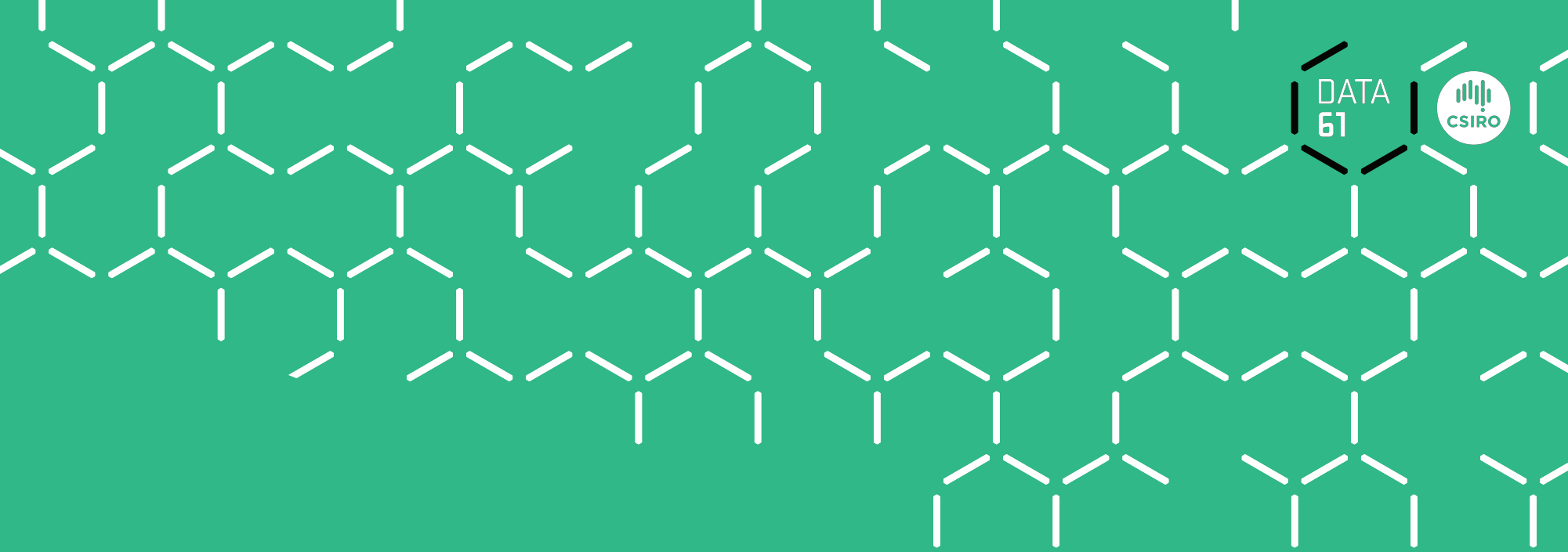
$$\mathsf{list\_rec}\ f_1\ f_2\ xs \equiv \mathsf{THE}\ y.\ (xs, y) \in \mathsf{list\_rel}\ f_1\ f_2$$

Automatic proof that set def indeed is total function
(the equations for list_rec are lemmas!)



Predefined Datatypes

nat is a datatype

datatype nat = 0 | Suc nat

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Functions on nat definable by primrec!

primrec
$f\ 0 = \ldots$
$f\ (\text{Suc}\ n) = \ldots\ f\ n\ \ldots$

Option

datatype 'a option = None | Some 'a

Important application:

'b ⇒ 'a option ∼ partial function:

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Example:
primrec lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option
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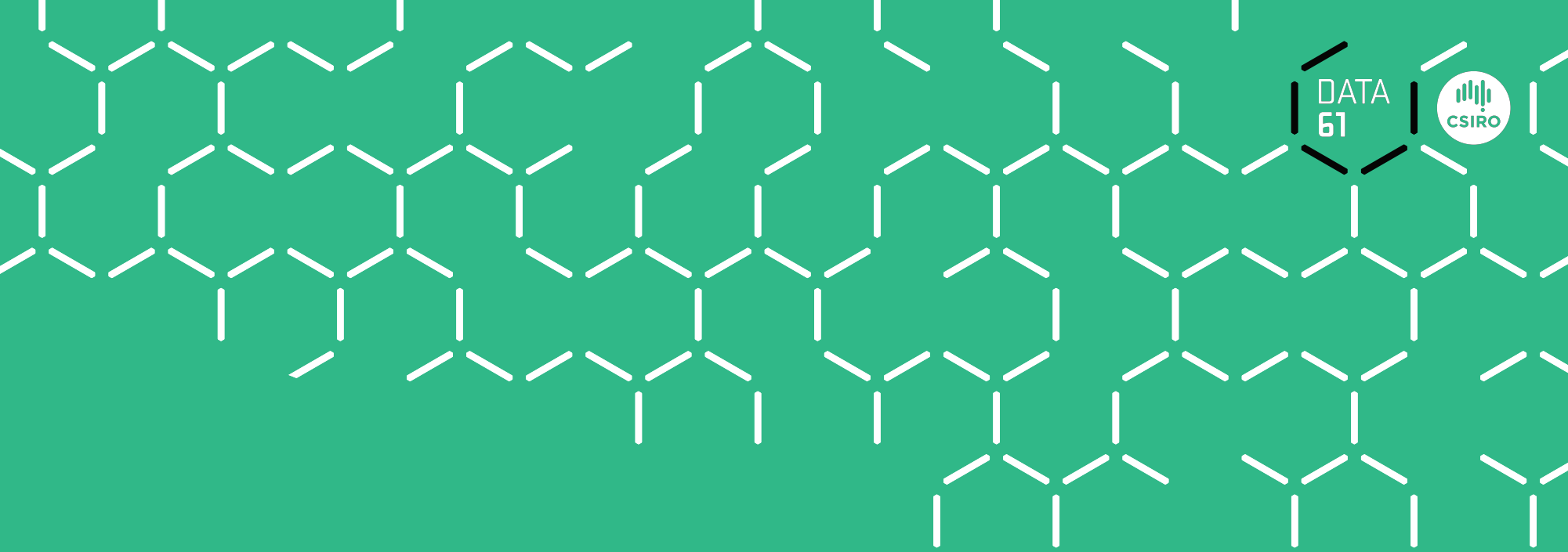
lookup k (x #xs) = (if fst x = k then Some (snd x) else lookup k xs)

Demo

primrec



Induction

Structural induction

$P\ xs$ holds for all lists xs if

- ➜ P Nil
- ➜ and for arbitrary x and xs, $P\ xs \Longrightarrow P\ (x\#xs)$

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  Induction theorem **list.induct:**
  $[\![P\ []; \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$

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  $[\![P\ []; \bigwedge a\ list.\ P\ list \Longrightarrow P\ (a\#list)]\!] \Longrightarrow P\ list$
- ➜ General proof method for induction: **(induct x)**
  - x must be a free variable in the first subgoal.
  - type of x must be a datatype.

Basic heuristics

Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

Example

A tail recursive list reverse:

primrec itrev :: 'a list ⇒ 'a list ⇒ 'a list
where
itrev [] *ys* = *ys* |

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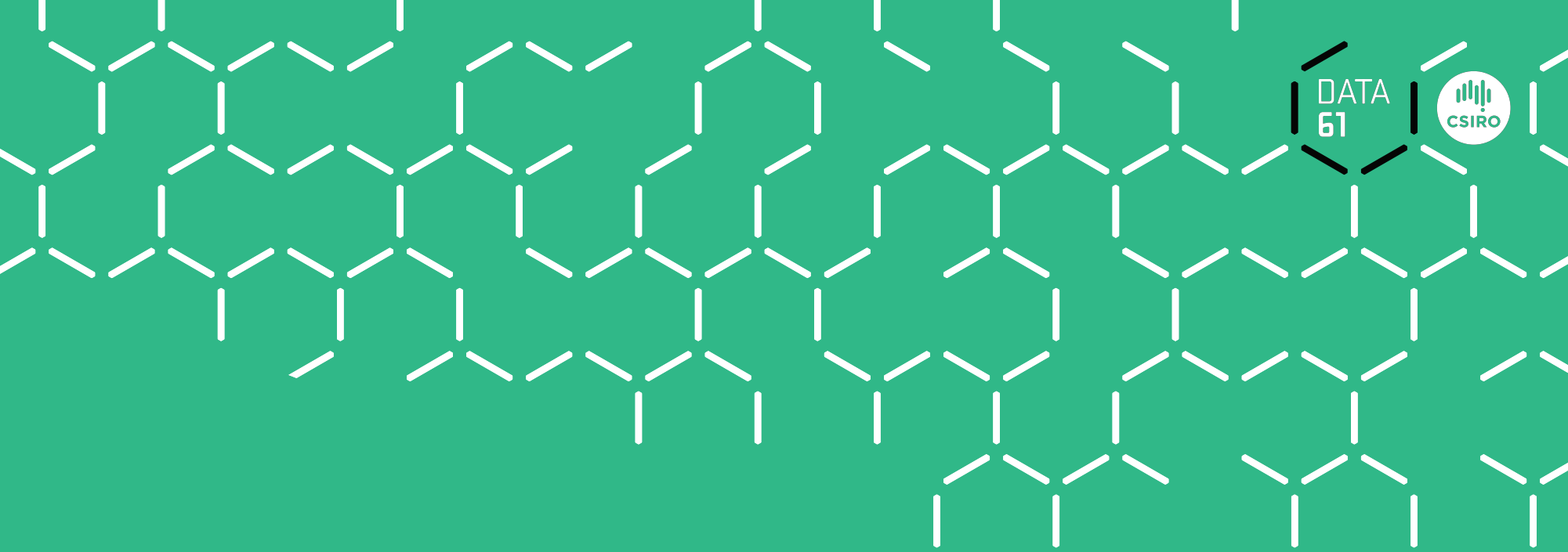
lemma itrev xs [] = rev xs



Demo

Proof Attempt

Generalisation

Replace constants by variables

lemma itrev xs ys = rev xs@ys

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Replace constants by variables

lemma itrev *xs ys* = rev *xs*@*ys*

Quantify free variables by $\forall$
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lemma itrev *xs ys* = rev *xs*@*ys*

Quantify free variables by ∀
(except the induction variable)

lemma ∀*ys*. itrev *xs ys* = rev *xs*@*ys*

Or: **apply (induct xs arbitrary: ys)**

We have seen today ...

- ➜ Datatypes

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- ➜ Primitive recursion
- ➜ Case distinction
- ➜ Structural Induction

Exercises

- ➜ define a primitive recursive function **lsum** :: nat list ⇒ nat that returns the sum of the elements in a list.
- ➜ show "$2 * \text{lsum}\ [0..<\mathit{Suc}\ n] = n * (n + 1)$"
- ➜ show "$\text{lsum}\ (\text{replicate}\ n\ a) = n * a$"
- ➜ define a function **lsumT** using a tail recursive version of listsum.
- ➜ show that the two functions are equivalent: $\text{lsum}\ xs = \text{lsumT}\ xs$

