DATA 61

COMP4161: Advanced Topics in Software Verification

Isar

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S2/2017

CSIRO

Content

- Intro & motivation, getting started [1]

- Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic [3[a]]
  - Term rewriting [4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction [6, 7]
  - Hoare logic, proofs about programs, C verification [8[b],9]
  - (mid-semester break)
  - Writing Automated Proof Methods [10]
  - Isar, codegen, typeclasses, locales [11[c],12]

[a] a1 due; [b] a2 due; [c] a3 due

Isar

A Language for Structured Proofs

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

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YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```


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or `by blast`

OK it's true. But WHY?

Motivation

WHY is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

Demo

Isar

apply scripts

➜ unreadable

Isar

apply scripts

➜ unreadable
➜ hard to maintain

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apply scripts		**What about..**	
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apply scripts	**What about..**
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No structure.	**Isar!**

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** simp
  $\vdots$
  have $formula_n$ **by** blast
  show $formula_{n+1}$ **by** ...
qed

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proves $formula_0 \Longrightarrow formula_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

Isar core syntax

$$\begin{array}{rcl} \text{proof} & = & \textbf{proof}\ [\text{method}]\ \text{statement}^*\ \textbf{qed} \\ & | & \textbf{by}\ \text{method} \end{array}$$

$$\text{method} = (\text{simp} \ldots) \mid (\text{blast} \ldots) \mid (\text{rule} \ldots) \mid \ldots$$

Isar core syntax

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\begin{array}{rcll}
\text{proof} & = & \textbf{proof}\ [\text{method}]\ \text{statement}^*\ \textbf{qed} & \\
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\\
\text{method} & = & (\text{simp}\ \ldots)\ |\ (\text{blast}\ \ldots)\ |\ (\text{rule}\ \ldots)\ |\ \ldots & \\
\\
\text{statement} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\
& | & \textbf{assume}\ \text{proposition} & (\Longrightarrow) \\
& | & [\textbf{from}\ \text{name}^+]\ (\textbf{have}\ |\ \textbf{show})\ \text{proposition proof} & \\
& | & \textbf{next} & \text{(separates subgoals)}
\end{array}
$$

Isar core syntax

proof = **proof** [method] statement* **qed**
    | **by** method

method = (simp . . .) | (blast . . .) | (rule . . .) | . . .

statement = **fix** variables ($\bigwedge$)
    | **assume** proposition ($\Longrightarrow$)
    | [**from** name$^+$] (**have** | **show**) proposition proof
    | **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$"
proof (rule conjI)

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lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$"
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➜ **proof** (<method>) applies method to the stated goal

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➜	**proof** (<method>)	applies method to the stated goal
➜	**proof**	applies a single rule that fits
➜	**proof** -	does nothing to the goal

How do I know what to Assume and Show?

Look at the proof state!

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- ➜ so we need 2 shows: **show** "A" and **show** "B"

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➜ We are allowed to **assume** A,
because A is in the assumptions of the proof state.

The Three Modes of Isar

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  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** **. . .**

Have

Can be used to make intermediate steps.

Example:

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Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"

Have

Can be used to make intermediate steps.

Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc } x$" **by** simp
  have B: "$1 + x = \text{Suc } x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

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Backward and Forward

Backward reasoning: $\ldots$ **have** "$A \wedge B$" **proof**

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Forward reasoning: $\ldots$

assume AB: "$A \wedge B$"
from AB **have** "$\ldots$" **proof**

Backward and Forward

Backward reasoning: ... **have** "$A \wedge B$" **proof**

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assume AB: "$A \wedge B$"
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Forward reasoning: ...

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from AB **have** "..." **proof**

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- first assumption of rule must unify with AB

General case: **from** A_1 ... A_n **have** R **proof**

- first n assumptions of rule must unify with A_1 ... A_n
- conclusion of rule must unify with R

Fix and Obtain

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Introduces new variables together with property

Demo

Fancy Abbreviations

this $=$ the previous fact proved or assumed

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this	=	the previous fact proved or assumed
then	=	**from** this

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Fancy Abbreviations

this	$=$	the previous fact proved or assumed
then	$=$	**from** this
thus	$=$	**then show**
hence	$=$	**then have**
with $A_1 \ldots A_n$	$=$	**from** $A_1 \ldots A_n$ this
?thesis	$=$	the last enclosing goal statement

Moreover and Ultimately

have X_1: P_1 . . .
have X_2: P_2 . . .
$\vdots$
have X_n: P_n . . .
from $X_1 \ldots X_n$ **show** . . .

Moreover and Ultimately

have X_1: P_1 ...
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have X_n: P_n ...
from $X_1 \dots X_n$ **show** ...

wastes lots of brain power
on names $X_1 \dots X_n$

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have P_1 ...
moreover have P_2 ...
$\vdots$
moreover have P_n ...
ultimately show ...

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General Case Distinctions

show *formula*
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  have $P_1 \vee P_2 \vee P_3$ $<$proof$>$
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  ultimately show ?thesis **by** blast
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  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }
stands for $P_1 \Longrightarrow P$

Mixing proof styles

from ...
have ...
  apply -   make incoming facts assumptions
  apply (...)
  ⋮
  apply (...)
  done

Datatypes in Isar

Datatype case distinction

proof (cases *term*)
  case Constructor_1
  ⋮
next
⋮
next
  case $(\text{Constructor}_k\ \vec{x})$
  $\cdots\ \vec{x}\ \cdots$
qed

Datatype case distinction

proof (cases *term*)
  case Constructor_1
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  $\cdots\ \vec{x}\ \cdots$
qed

$$\textbf{case}\ (\text{Constructor}_i\ \vec{x}) \quad \equiv$$
$$\textbf{fix}\ \vec{x}\ \textbf{assume}\ \text{Constructor}_i : \text{"}\mathit{term} = \text{Constructor}_i\ \vec{x}\text{"}$$

Structural induction for nat

show $P\ n$		
proof (induct n)		
case 0	$\equiv$	**let** $?case = P\ 0$
...		
show $?case$		
next		
case (Suc n)	$\equiv$	**fix** n **assume** Suc: $P\ n$
...		**let** $?case = P\ (\mathrm{Suc}\ n)$
... n ...		
show $?case$		
qed		

Structural induction: $\Longrightarrow$ and $\bigwedge$

show "$\bigwedge x.\ A\ n \Longrightarrow P\ n$"
proof (induct n)
  case 0 $\equiv$ **fix** x **assume** 0: "$A\ 0$"
   let $?case$ = "$P\ 0$"
  ...
  show $?case$
next
  case (Suc n) $\equiv$ **fix** n and x
   assume Suc: "$\bigwedge x.\ A\ n \Longrightarrow P\ n$"
     "$A\ (\text{Suc}\ n)$"
   let $?case$ = "$P\ (\text{Suc}\ n)$"
  ...
  $\cdots\ n\ \cdots$
  ...
  show $?case$
qed

Demo: Datatypes in Isar

Calculational Reasoning

The Goal

Prove:
$x \cdot x^{-1} = 1$

using:

using:	assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
	left_inv:	$x^{-1} \cdot x = 1$
	left_one:	$1 \cdot x = x$

The Goal

Prove:

$$
\begin{aligned}
x \cdot x^{-1} &= 1 \cdot (x \cdot x^{-1}) \\
\ldots &= 1 \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (x^{-1} \cdot x) \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot 1 \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (1 \cdot x^{-1}) \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \\
\ldots &= 1
\end{aligned}
$$

assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
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\ldots &= 1
\end{aligned}
$$

assoc: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

left_inv: $x^{-1} \cdot x = 1$

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Can we do this in Isabelle?

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\end{aligned}
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assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
left_one:	$1 \cdot x = x$

Can we do this in Isabelle?

➜ Simplifier: too eager

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Prove:

$$
\begin{aligned}
x \cdot x^{-1} &= 1 \cdot (x \cdot x^{-1}) \\
\ldots &= 1 \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \cdot x \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (x^{-1} \cdot x) \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot 1 \cdot x^{-1} \\
\ldots &= (x^{-1})^{-1} \cdot (1 \cdot x^{-1}) \\
\ldots &= (x^{-1})^{-1} \cdot x^{-1} \\
\ldots &= 1
\end{aligned}
$$

assoc:	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
left_inv:	$x^{-1} \cdot x = 1$
left_one:	$1 \cdot x = x$

Can we do this in Isabelle?

- ➜ Simplifier: too eager
- ➜ Manual: difficult in apply style

The Goal

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Can we do this in Isabelle?

- ➜ Simplifier: too eager
- ➜ Manual: difficult in apply style
- ➜ Isar: with the methods we know, too verbose

Chains of equations

The Problem

$$\begin{array}{lcl} a & = & b \\ \ldots & = & c \\ \ldots & = & d \end{array}$$

shows $a = d$ by transitivity of $=$

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Each step usually nontrivial (requires own subproof)

Chains of equations

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Solution in Isar:

➜ Keywords **also** and **finally** to delimit steps

➜ **...**: predefined schematic term variable,
refers to right hand side of last expression

➜ Automatic use of transitivity rules to connect steps

also/finally

have "$t_0 = t_1$" [proof]
also

also/finally

have "$t_0 = t_1$" [proof]
also

calculation register
"$t_0 = t_1$"

also/finally

have "$t_0 = t_1$" [proof]
also
have "$\ldots = t_2$" [proof]

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"$t_0 = t_1$"

"$t_0 = t_2$"

also/finally

have "$t_0 = t_1$" [proof]
also
have "$\ldots = t_2$" [proof]
also
$\vdots$
also

calculation register
"$t_0 = t_1$"

"$t_0 = t_2$"
$\vdots$
"$t_0 = t_{n-1}$"

also/finally

have $"t_0 = t_1"$ [proof]
also
have $"\ldots = t_2"$ [proof]
also
$\vdots$
also
have $"\cdots = t_n"$ [proof]

calculation register
$"t_0 = t_1"$

$"t_0 = t_2"$
$\vdots$
$"t_0 = t_{n-1}"$

also/finally

have "$t_0 = t_1$" [proof]
also
have "$\ldots = t_2$" [proof]
also
$\vdots$
also
have "$\cdots = t_n$" [proof]
finally

calculation register
"$t_0 = t_1$"

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$\vdots$
"$t_0 = t_{n-1}$"

$t_0 = t_n$

also/finally

have "$t_0 = t_1$" [proof]
also
have "$\ldots = t_2$" [proof]
also
$\vdots$
also
have "$\cdots = t_n$" [proof]
finally
show P
— 'finally' pipes fact "$t_0 = t_n$" into the proof

calculation register
"$t_0 = t_1$"

"$t_0 = t_2$"
$\vdots$
"$t_0 = t_{n-1}$"

$t_0 = t_n$

More about also

➜ Works for all combinations of $=$, $\leq$ and $<$.

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- Works for all combinations of $=$, $\leq$ and $<$.
- Uses all rules declared as `[trans]`.
- To view all combinations: `print_trans_rules`

Designing [trans] Rules

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- ➜ More general form: $[\![P\ \ l_1\ \ r_1; Q\ \ r_1\ \ r_2; A]\!] \Longrightarrow C\ \ l_1\ \ r_2$

Examples:

Designing [trans] Rules

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have " $\ldots \odot r_2$" [proof]
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Examples:

- pure transitivity: $[\![a = b; b = c]\!] \Longrightarrow a = c$

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Examples:

- ➜ pure transitivity: $[\![a = b; b = c]\!] \Longrightarrow a = c$
- ➜ mixed: $[\![a \leq b; b < c]\!] \Longrightarrow a < c$

Designing [trans] Rules

$$\begin{array}{l}\textbf{have} = \text{"}l_1 \odot r_1\text{"}\ \text{[proof]}\\ \textbf{also}\\ \textbf{have}\ \text{"}\ldots \odot r_2\text{"}\ \text{[proof]}\\ \textbf{also}\end{array}$$

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Examples:

- pure transitivity: $[\![a = b; b = c]\!] \Longrightarrow a = c$
- mixed: $[\![a \leq b; b < c]\!] \Longrightarrow a < c$
- substitution: $[\![P\ \ a; a = b]\!] \Longrightarrow P\ \ b$

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- mixed: $[\![a \leq b; b < c]\!] \Longrightarrow a < c$
- substitution: $[\![P\ a; a = b]\!] \Longrightarrow P\ b$
- antisymmetry: $[\![a < b; b < a]\!] \Longrightarrow \mathit{False}$
- monotonicity: $[\![a = f\ b; b < c; \bigwedge x\ y.\ x < y \Longrightarrow f\ x < f\ y]\!] \Longrightarrow a < f\ c$

Demo