COMP4161: Advanced Topics in Software Verification

HOL

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction, datatype induction, primitive recursion [4,5]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Hoare logic, proofs about programs, invariants [8]
- C verification [9,10]
- Practice, questions, examp prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

More on Automation

Last time: safe and unsafe, heuristics: use safe before unsafe

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Example:

declare attribute globally	**declare** conjI [intro!] allE [elim]
remove attribute globally	**declare** allE [rule del]
use locally	**apply** (blast intro: someI)
delete locally	**apply** (blast del: conjI)

Demo: Automation

Exercises

- derive the classical contradiction rule $(\neg P \Longrightarrow \mathit{False}) \Longrightarrow P$ in Isabelle
- define **nor** and **nand** in Isabelle
- show $\quad$ nor $x\ x\ =\ $ nand $x\ x$
- derive safe intro and elim rules for them
- use these in an automated proof of $\quad$ nor $x\ x\ =\ $ nand $x\ x$

Defining Higher Order Logic

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➜ **Higher Order Logic:**
  - quantification over everything, including predicates
  - consistency by types
  - formula $=$ term of type bool
  - definition built on $\lambda^{\rightarrow}$ with certain default types and constants

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Default types:

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bool

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bool $\quad\quad\quad\quad$ $\_ \Rightarrow \_$

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One approach: $\forall :: \mathit{var} \Rightarrow \mathit{term} \Rightarrow \mathit{bool}$
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So: Use λ to encode all other binders.

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HOAS	**usual syntax**
ALL $(\lambda x.\ x = 2)$	$\forall x.\ x = 2$
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Isabelle can translate usual binder syntax into HOAS.

Side Track: Syntax Declarations

- ➜ **mixfix:**
  consts drvbl :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, _ ⊢ _")
  Legal syntax now: $\Gamma, \Pi \vdash F$

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  $c :: (\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder "B" $< p >$)
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More in Isabelle/Isar Reference Manual (7.2)

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The Axioms of HOL

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t \quad P\ s}{P\ t}\ \text{subst} \qquad \frac{\bigwedge x.\ f\ x = g\ x}{(\lambda x.\ f\ x) = (\lambda x.\ g\ x)}\ \text{ext}$$

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- ➜ 3 basic types
- ➜ 9 axioms

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Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y}\ \text{eq\_reflection} \qquad \frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Demo:
The Definitions in Isabelle

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Convenient for deriving rules: **named assumptions in lemmas**

lemma [*name* :]
assumes [*name*$_1$:] "$< prop >_1$"
assumes [*name*$_2$:] "$< prop >_2$"
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shows "$< prop >$" $< proof >$

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shows "$< prop >$" $< proof >$

proves: $[\![< prop >_1; \; < prop >_2; \; \ldots]\!] \Longrightarrow < prop >$

True

consts True :: *bool*
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Intuition:
right hand side is always true

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$$\frac{}{\text{True}}\ \text{TrueI}$$

Proof:

$$\cfrac{\overline{(\lambda x :: \mathit{bool}.\ x) = (\lambda x.\ x)}\ \text{refl}}{\text{True}}\ \text{unfold True\_def}$$

Demo

Universal Quantifier

consts ALL :: $(\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$
ALL $P \quad \equiv \quad P = (\lambda x.\ \text{True})$

Intuition:

- ➜ ALL P is Higher Order Abstract Syntax for $\forall x.\ P\ x$.
- ➜ P is a function that takes an x and yields a truth value.
- ➜ ALL P should be true iff P yields true for all x, i.e.
  if it is equivalent to the function $\lambda x.$ True.

Proof Rules:

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

Proof: Isabelle Demo

False

consts False :: *bool*
False $\equiv$ $\forall P.P$

Intuition:
Everything can be derived from *False*.

Proof Rules:

$$\frac{\text{False}}{P}\ \text{FalseE} \qquad \frac{}{\text{True} \neq \text{False}}$$

Proof: Isabelle Demo

Negation

consts Not :: $bool \Rightarrow bool$ $(\neg\ \_)$
$\neg P \equiv P \longrightarrow \text{False}$

Intuition:
Try $P = \mathit{True}$ and $P = \mathit{False}$ and the traditional truth table for $\longrightarrow$.

Proof Rules:

$$\frac{A \Longrightarrow \mathit{False}}{\neg A}\ \text{notI} \qquad \frac{\neg A \quad A}{P}\ \text{notE}$$

Proof: Isabelle Demo

Existential Quantifier

consts EX :: $(\alpha \Rightarrow bool) \Rightarrow bool$

$\text{EX}\ P \quad \equiv \quad \forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q$

Intuition:

- ➜ EX P is HOAS for $\exists x.\ P\ x.$ (like $\forall$)
- ➜ Right hand side is characterization of $\exists$ with $\forall$ and $\longrightarrow$
- ➜ Note that inner $\forall$ binds wide: $(\forall x.\ P\ x \longrightarrow Q)$
- ➜ Remember lemma from last time: $(\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$

Proof Rules:

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

Proof: Isabelle Demo

Conjunction

consts And :: $\mathit{bool} \Rightarrow \mathit{bool} \Rightarrow \mathit{bool}$ $(\_ \wedge \_)$
$P \wedge Q \equiv \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

Intuition:

- ➜ Mirrors proof rules for $\wedge$
- ➜ Try truth table for P, Q, and R

Proof Rules:

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

Proof: Isabelle Demo

Disjunction

consts Or :: $\mathit{bool} \Rightarrow \mathit{bool} \Rightarrow \mathit{bool}$ $(\_ \lor \_)$
$P \lor Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

Intuition:

- ➜ Mirrors proof rules for $\lor$ (case distinction)
- ➜ Try truth table for P, Q, and R

Proof Rules:

$$\frac{A}{A \lor B} \quad \frac{B}{A \lor B}\ \text{disjI1/2} \qquad \frac{A \lor B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

Proof: Isabelle Demo

If-Then-Else

consts If :: *bool* $\Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)
If $P\ x\ y\ \equiv$ SOME $z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- for $P =$ True, right hand side collapses to SOME $z.\ z = x$
- for $P =$ False, right hand side collapses to SOME $z.\ z = y$

Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s}\ \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t}\ \text{ifFalse}$$

Proof: Isabelle Demo

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