



COMP4161: Advanced Topics in Software Verification



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Content

→ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3^a]
- Term rewriting [3,4]

→ Proof & Specification Techniques

- Inductively defined sets, rule induction, datatype induction, primitive recursion [4,5]
- General recursive functions, termination proofs [7^b]
- Proof automation, Hoare logic, proofs about programs, invariants [8]
- C verification [9,10]
- Practice, questions, exam prep [10^c]

^aa1 due; ^ba2 due; ^ca3 due

Last Time on HOL

→ Defining HOL

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- More automation

Term Rewriting

The Problem

Given a set of equations

$$l_1 = r_1$$

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$$l_n = r_n$$

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Applications in:

- **Mathematics** (algebra, group theory, etc)
- **Functional Programming** (model of execution)
- **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$l_1 \longrightarrow r_1$$

$$l_2 \longrightarrow r_2$$

\vdots

$$l_n \longrightarrow r_n$$

decide $l = r$ by deciding $l \overset{*}{\longleftarrow} r$

Arrow Cheat Sheet

$$\xrightarrow{0} = \{(x, y) | x = y\} \quad \text{identity}$$

Arrow Cheat Sheet

$$\begin{aligned} \xrightarrow{0} &= \{(x, y) \mid x = y\} && \text{identity} \\ \xrightarrow{n+1} &= \xrightarrow{n} \circ \longrightarrow && \text{n+1 fold composition} \end{aligned}$$

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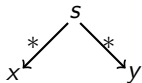
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Fact: \longrightarrow is Church-Rosser iff it is confluent.

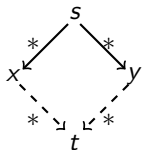
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Problem:

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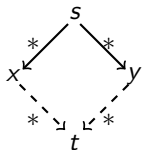


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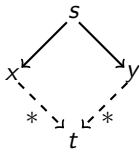


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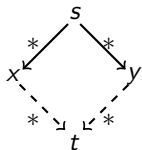
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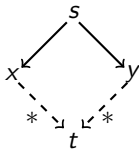


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Fact: local confluence and termination \implies confluence

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- ② $<_r$ is well founded, because $<$ is well founded on \mathbb{N}

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True for most orders that don't treat certain parts of terms as special cases.

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We show that the rewrite system defined by these rules is terminating.

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- $s <_i t \equiv \text{num_imps } s < \text{num_imps } t$ and
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Then $<_i$ and $<_n$ are both well-founded orders (since both return nats).

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 $<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

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$$\text{osize}' c \quad x = 2^x$$

$$\text{osize}' (\neg P) \quad x = \text{osize}' P (x + 1)$$

$$\text{osize}' (P \wedge Q) \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1))$$

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$$\text{osize } P \quad = \text{osize}' P 0$$

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The other rules decrease the depth of the things osize counts, so decrease osize.

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- (almost) blindly from left to right
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confluence: not guaranteed
(result may depend on which rule is used first)

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- Adding/deleting equations locally:
apply (simp add: <rules>) and **apply** (simp del: <rules>)
- Using only the specified set of equations:
apply (simp only: <rules>)

Demo

We have seen today...

→ Equations and Term Rewriting

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- Equations and Term Rewriting
- Confluence and Termination of reduction systems

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- Confluence and Termination of reduction systems
- Term Rewriting in Isabelle

Exercises

- Show, via a pen-and-paper proof, that the `osize` function is monotonic with respect to the structure of terms from that example.