COMP4161: Advanced Topics in Software Verification

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Content

➜ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction, datatype induction, primitive recursion [4,5]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Hoare logic, proofs about programs, invariants [8]
- C verification [9,10]
- Practice, questions, examp prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

Last Time

➜ Equations and Term Rewriting

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is **applicable** to term $t[s]$ with σ if

- ➜ $\sigma\ l = s$ and
- ➜ $\sigma\ P_1, \dots, \sigma\ P_n$ are provable by rewriting.

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simp	**use and simplify** assumptions
(simp (no_asm))	**ignore** assumptions
(simp (no_asm_use))	**simplify**, but do **not use** assumptions
(simp (no_asm_simp))	**use**, but do **not simplify** assumptions

Preprocessing

Preprocessing (recursive) for maximal simplification power:

$$\begin{array}{rcl} \neg A & \mapsto & A = \mathit{False} \\ A \longrightarrow B & \mapsto & A \Longrightarrow B \\ A \wedge B & \mapsto & A,\ B \\ \forall x.\ A\ x & \mapsto & A\ ?x \\ A & \mapsto & A = \mathit{True} \end{array}$$

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Demo

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Similar for any data type t: **t.split**

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if_cong: $[\![b = c; c \Longrightarrow x = u; \neg c \Longrightarrow y = v]\!] \Longrightarrow$
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- ➜ use locally with e.g. **apply** (simp cong: <rule>)

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For types nat, int etc:

- lemmas **add_ac** sort any sum $(+)$
- lemmas **mult_ac** sort any product $(*)$

Example: **apply** (simp add: add_ac) yields
$(b + c) + a \leadsto \cdots \leadsto a + (b + c)$

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If these 3 rules are present for an AC operator
Isabelle will order terms correctly

Demo

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Rules: (1) $f\ x \longrightarrow a$ (2) $g\ y \longrightarrow b$ (3) $f\ (g\ z) \longrightarrow b$
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$$(1)+(3) \qquad \{x \mapsto g\ z\} \qquad a \overset{(1)}{\longleftarrow} f\ (g\ z) \overset{(3)}{\longrightarrow} b$$

$$(3)+(2) \qquad \{z \mapsto y\} \qquad b \overset{(3)}{\longleftarrow} f\ (g\ y) \overset{(2)}{\longrightarrow} f\ b$$

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This is the main idea of the Knuth-Bendix completion algorithm.

Demo: Waldmeister

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Application: functional programming languages

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