COMP9020 Lectures 2-3
Session 2, 2017
Functions and Relations
Quiz 1 due tomorrow 23:59

Quiz 2 available tonight, due 17 August 15:00

Assignment 1 available Saturday, due 25 August 23:59
  Lateness policy: -10% per 12 hours or part thereof
Lecture 1 recap: Numbers

- Floor, \[
\lfloor \cdot \rfloor
\]
- Ceiling, \[
\lceil \cdot \rceil
\]
- Divides relation, \[
|
\]
- GCD and LCM and how to compute them
- Absolute value: \[
|\cdot|
\]
Lecture 1 recap: Sets

- $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{P}$
- Union $\cup$, intersection $\cap$, complement $A^c$, set difference $\setminus$, symmetric difference $\oplus$
- Power set, Pow, subset $\subseteq$ and $\subset$
- Cartesian product
- Empty set
Solving set theory problems

1. Use a Venn diagram
2. Use set operations and their properties

Examples

- \((A \setminus B) \setminus C = A \cap B^c \cap C^c\);
- \(A \setminus (B \setminus C) = (A \cap B^c) \cup (A \cap C)\).

NB

\(A \setminus B \overset{\text{def}}{=} A \cap B^c\)
Lecture 1 recap: Alphabets

**Alphabet:** Finite set of **symbols**, e.g. $\Sigma = \{a, b\}$

**Word:** Finite **string** of symbols, e.g. $abba$, $\lambda$

**Language:** Set of words, e.g. $\Sigma^*$, $\Sigma^+$, $\Sigma^{\leq 3} = \{w : \text{length}(w) \leq 3\}$.
COMP9020 Lectures 2-3
Session 2, 2017
Functions and Relations

- Textbook - Ch. 3, Sec. 3.1, 3.3–3.4; Ch. 11, Sec. 11.1–11.2
- Problem sets 2 and 3
- Supplementary Exercises Ch. 3 and 11 (R & W)
Recall: Functions

Recall:
\[ f : S \rightarrow T \]

\( S \) — **domain** of \( f \), symbol: \( \text{Dom}(f) \)

\( T \) — **codomain** of \( f \), symbol: \( \text{Codom}(f) \)

\{ \( f(x) : x \in \text{Dom}(f) \) \} — **image** of \( f \), symbol: \( \text{Im}(f) \)

Function composition, \( f \circ g : x \mapsto f(g(x)) \)
Properties of Functions

Function is called **surjective** or **onto** if every element of the codomain is mapped to by at least one $x$ in the domain, i.e.

$$\text{Im}(f) = T$$

**Examples (of functions that are surjective)**

- $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(x) \mapsto x$
- Floor, ceiling

**Examples (of functions that are not surjective)**

- $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(x) \mapsto x^2$
- $f : \{a, \ldots, z\}^* \rightarrow \{a, \ldots, z\}^*$ with $f(\omega) \mapsto a\omega e$
Injective Functions

Function is called injective or 1–1 (one-to-one) if different $x$ implies different $f(x)$, i.e.

$$f(x) = f(y) \Rightarrow x = y$$

Examples (of functions that are injective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- set complement (for a fixed universe)

Examples (of functions that are not injective)

- absolute value, floor, ceiling
- length of a word

Function is bijective if it is both surjective and injective.
Inverse Functions

**Inverse** function — \( f^{-1} : T \rightarrow S \);
for a given \( f : S \rightarrow T \) exists exactly
when \( f \) is bijective.

Image of a subdomain \( A \) under a function

\[
f(A) = \{ f(s) : s \in A \} = \{ t \in T : t = f(s) \text{ for some } s \in A \}
\]

**Inverse image** — \( f^{-1}(B) = \{ s \in S : f(s) \in B \} \subseteq S \);
it is defined for every \( f \)

If \( f^{-1} \) exists then \( f^{-1}(B) = f^{-1}(B) \)

\[
f(\emptyset) = \emptyset, f^{-1}(\emptyset) = \emptyset
\]
Examples

1.7.5 \( f \) and \( g \) are ‘shift’ functions \( \mathbb{N} \rightarrow \mathbb{N} \) defined by
\( f(n) = n + 1 \), and \( g(n) = \max(0, n - 1) \)

(c) Is \( f \) injective? surjective?
(d) Is \( g \) injective? surjective?
(e) Do \( f \) and \( g \) commute, i.e. \( \forall n \ ( (f \circ g)(n) = (g \circ f)(n) ) \)?
Examples

1.7.5 \( f \) and \( g \) are ‘shift’ functions \( \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(n) = n + 1 \), and \( g(n) = \max(0, n - 1) \)

(c) \( f \) is injective, not surjective: \( f(\mathbb{N}) = \mathbb{N} \setminus \{0\} = \mathbb{P} \)

(d) \( g \) is surjective, not injective: \( g(0) = g(1) \)

(e) \( f \) and \( g \) do not commute:
\( g \circ f : n \mapsto (n + 1) - 1 = n \), thus \( g \circ f = \text{Id}_{\mathbb{N}} \)
\( f \circ g : 0 \mapsto 1 \), hence \( f \circ g \neq \text{Id}_{\mathbb{N}} \)

NB

\( f \circ g \) is the identity when restricted to \( \mathbb{P} \)
NB

For a **finite** set $S$ and $f : S \rightarrow S$ the properties

1. **surjective**, and
2. **injective**

are equivalent. *(Proof suggestion?)*
Examples

1.7.6 \( \Sigma = \{a, b, c\} \)

(c) Is \( \text{length} : \Sigma^* \rightarrow \mathbb{N} \) surjective?

(d) \( \text{length}^{-1}(2) = \)

Examples

1.7.12 Verify that \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \) defined by
\( f(x, y) = (x + y, x - y) \) is invertible.
Examples

1.7.6 \( \Sigma = \{a, b, c\} \)

(c) Is \( \text{length} : \Sigma^* \longrightarrow \mathbb{N} \) surjective?
Yes: \( \text{length}^{-1}(\{ n \}) = \Sigma^n \neq \emptyset \)

(d) \( \text{length}^{-1}(2) = \{aa, ab, ac, bb, \ldots, cc\} \)

Examples

1.7.12 Verify that \( f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R} \) defined by
\( f(x, y) = (x + y, x - y) \) is invertible.
The inverse is \( f^{-1}(a, b) = \left( \frac{a+b}{2}, \frac{a-b}{2} \right) \); substituting shows that \( f \circ f^{-1} = \text{Id}_{\mathbb{R} \times \mathbb{R}} \)
1.8.16 \( \Sigma = \{ a, b \} \); relate it to \( \Sigma^* \)
(a) Is there a surjection \( \Sigma \rightarrow \Sigma^* \)?
(b) Is there a surjection \( \Sigma^* \rightarrow \Sigma \)?
1.8.16 \( \Sigma = \{a, b\} \); relate it to \( \Sigma^* \)

(a) Is there a surjection \( \Sigma \rightarrow \Sigma^* \)? No: \(|\Sigma| = 2, |\Sigma^*| = \infty \).

(b) Is there a surjection \( \Sigma^* \rightarrow \Sigma \)? Yes, eg \( f(\omega) = a \) when \( \text{length}(\omega) \) is odd, \( f(\omega) = b \) when \( \text{length}(\omega) \) is even.

The following is not completely correct \( f : \omega \mapsto \langle \text{first letter of } \omega \rangle \)

Reason: \( f(\lambda) \) is not defined.
Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are related. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

In general, relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.
An **n-ary relation** is a subset of the cartesian product of $n$ sets.

$$R \subseteq S_1 \times S_2 \times \ldots \times S_n$$

$x \in R \Rightarrow x = (x_1, x_2, \ldots, x_n)$ where each $x_i \in S_i$

If $n = 2$ we have a **binary** relation $\mathcal{R} \subseteq S \times T$.

(mostly we consider binary relations)

equivalent notations: $(x_1, x_2, \ldots, x_n) \in R \iff R(x_1, x_2, \ldots, x_n)$

for binary relations: $(x, y) \in R \iff R(x, y) \iff xRy$. 
Example (course enrolments)

\[ S = \text{set of CSE students} \]
\[(S \text{ can be a subset of the set of all students)}\]

\[ C = \text{set of CSE courses} \]
\[(\text{likewise})\]

\[ E = \text{enrolments} = \{ (s, c) : s \text{ takes } c \} \]

\[ E \subseteq S \times C \]

In practice, almost always there are various ‘onto’ (nonemptiness) and \(1\)–\(1\) (uniqueness) constraints on database relations.
Example (class schedule)

$C = \text{CSE courses}$

$T = \text{starting time (hour & day)}$

$R = \text{lecture rooms}$

$S = \text{schedule} =$

$$\{ (c, t, r) : c \text{ is at } t \text{ in } r \} \subseteq C \times T \times R$$

Example (sport stats)

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$
Applications

Relations are ubiquitous in Computer Science

- Databases are collections of relations
- Common data structures (e.g. graphs) are relations
- Any ordering is a relation
- Functions/procedures/programs compute relations between their input and output

Relations are therefore used in most problem specifications and to describe formal properties of programs. For this reason, studying relations and their properties helps with formalisation, implementation and verification of programs.
Relations can be defined linking $k \geq 1$ domains $D_1, \ldots, D_k$ simultaneously. In database situations one also allows for unary ($n = 1$) relations. Most common are binary relations

$$\mathcal{R} \subseteq S \times T; \quad \mathcal{R} = \{(s, t) : \text{“some property that links } s, t\text{”}\}$$

For related $s, t$ we can write $(s, t) \in \mathcal{R}$ or $s \mathcal{R} t$; for unrelated items either $(s, t) \notin \mathcal{R}$ or $s \not\mathcal{R} t$.

$\mathcal{R}$ can be defined by

- explicit enumeration of interrelated $k$-tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $D_1 \times D_2 \times \ldots \times D_k$;
- construction from other relations.
Functions as Relations

Any function $f : S \rightarrow T$ can be viewed as a binary relation

$$\{ (s, f(s)) : s \in S \} \subseteq S \times T$$

If a subset of $S \times T$ corresponds to a function, it must satisfy certain conditions w.r.t. $S$ and $T$ (which?)
A binary relation, say $\mathcal{R} \subseteq S \times T$, can be presented as a matrix with rows enumerated by (the elements of) $S$ and the columns by $T$; eg. for $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3, t_4\}$ we may have

$$
\begin{bmatrix}
\bullet & \circ & \bullet & \bullet \\
\circ & \bullet & \bullet & \bullet \\
\bullet & \bullet & \circ & \circ \\
\end{bmatrix}
$$
3.1.2(e) Write the following relation on $A = \{0, 1, 2\}$ as a matrix.

$$(m, n) \in R \text{ if } m \cdot n = m$$

$$
\begin{bmatrix}
0 & 1 & 2 \\
0 & \bullet & \bullet & \bullet \\
1 & \circ & \bullet & \circ \\
2 & \circ & \bullet & \circ
\end{bmatrix}
$$
Particularly important are binary relationships between the elements of the same set. We say that ‘$\mathcal{R}$ is a relation on $S$’ if

$$\mathcal{R} \subseteq S \times S$$
(all w.r.t. set $S$)

**Identity**  (diagonal, equality) \[ E = \{ (x, x) : x \in S \} \]

**Empty**  $\emptyset$

**Universal**  $U = S \times S$
### Important Properties of Binary Relations $\mathcal{R} \subseteq S \times S$

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R) reflexive</td>
<td>$(x, x) \in \mathcal{R}$ $\forall x \in S$</td>
</tr>
<tr>
<td>(AR) antireflexive</td>
<td>$(x, x) \notin \mathcal{R}$ $\forall x \in S$</td>
</tr>
<tr>
<td>(S) symmetric</td>
<td>$(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$ $\forall x, y \in S$</td>
</tr>
<tr>
<td>(AS) antisymmetric</td>
<td>$(x, y), (y, x) \in \mathcal{R} \Rightarrow x = y$ $\forall x, y \in S$</td>
</tr>
<tr>
<td>(T) transitive</td>
<td>$(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$ $\forall x, y, z \in S$</td>
</tr>
</tbody>
</table>

**NB**

An object, notion etc. is considered to satisfy a property if none of its instances violates any defining statement of that property.
Examples

(R) reflexive \((x, x) \in \mathcal{R}\) for all \(x \in S\) 

(AR) antireflexive \((x, x) \not\in \mathcal{R}\)

(S) symmetric \((x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}\)

(AS) antisymmetric \((x, y), (y, x) \in \mathcal{R} \Rightarrow x = y\)

(T) transitive \((x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}\)
3.1.1 The following relations are on $S = \{1, 2, 3\}$. Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$ (AR) and (S)

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$ (S)

3.1.2(b) $(m, n) \in R$ if $m < n$ (AR), (AS), (T)
A relation can be both symmetric and antisymmetric. Namely, when $R$ consists only of some pairs $(x, x)$, $x \in S$.
A relation cannot be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

**NB**

\[
\begin{align*}
\text{nonreflexive} & \quad \text{is not the same as} \quad \text{antireflexive/irreflexive} \\
\text{nonsymmetric} & \quad \text{antisymmetric}
\end{align*}
\]
Most important kinds of relations on $S$

- **total order**
  - $\begin{bmatrix} \circ & \circ & \circ \\
  \circ & \circ & \circ \\
  \circ & \circ & \circ \end{bmatrix}$

- **partial order**
  - $\begin{bmatrix} \circ & \circ & \circ \\
  \circ & \circ & \circ \\
  \circ & \circ & \circ \end{bmatrix}$, $\begin{bmatrix} \circ & \circ & \circ \\
  \circ & \circ & \circ \\
  \circ & \circ & \circ \end{bmatrix}$

- **equivalence**
  - $\begin{bmatrix} \circ & \circ & \circ \\
  \circ & \circ & \circ \\
  \circ & \circ & \circ \end{bmatrix}$

- **identity**
  - $\begin{bmatrix} \circ & \circ & \circ \\
  \circ & \circ & \circ \\
  \circ & \circ & \circ \end{bmatrix}$

**NB**

*Some of those are special cases of the others, eg. ‘total order’ of a ‘partial order’, ‘identity’ of an ‘equivalence’.*
Relation $\mathcal{R}$ as Correspondence From $S$ to $T$

$$\mathcal{R}(A) \overset{\text{def}}{=} \{ t \in T : (s, t) \in \mathcal{R} \text{ for some } s \in A \subseteq S \}$$

$$\mathcal{R}^\leftarrow(B) \overset{\text{def}}{=} \{ s \in S : (s, t) \in \mathcal{R} \text{ for some } t \in B \subseteq T \}$$

Converse relation $\mathcal{R}^\leftarrow$

$$\mathcal{R}^\leftarrow = \{(t, s) \in T \times S : (s, t) \in \mathcal{R}\}$$

Note that $\mathcal{R}^\leftarrow \subseteq T \times S$.

Observe that $(\mathcal{R}^\leftarrow)^\leftarrow = \mathcal{R}$. 
NB

Viewed this way \( R \) becomes a function from \( \text{Pow}(S) \) to \( \text{Pow}(T) \). However, not every \( g : \text{Pow}(S) \to \text{Pow}(T) \) can be matched to a relation.

(Why? Using a small domain like \( S = \{a, b\} \), provide an example of a function \( g : \text{Pow}(S) \to \text{Pow}(S) \) which does not correspond to any relation on \( S \)!. Can you even do it with \( S' = \{a\} \)?)

NB

The order of axes – \( S \) and \( T \) – is important. For \( R \subseteq S \times S \), its converse \( R^\leftarrow \) is usually quite different from \( R \).

Example: divisibility relation on \( \mathbb{P} \)

\[
D \overset{\text{def}}{=} \{ (p, q) : p \mid q \} = \{(1, 1), (1, 2), \ldots, (2, 2), (2, 4), \ldots\}
\]

\[
D^\leftarrow = \{ (p, q) : p \in q^{\mathbb{P}} \}
\]

\[
= \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), \ldots\}
\]

For every \( n \in \mathbb{P} \), \( D(\{n\}) \) is infinite, \( D^\leftarrow(\{n\}) \) is finite.
Question

$f$ is a relation; when is it a function?
Question

$f \leftarrow$ is a relation; when is it a function?

Answer

When $f$ is 1-1 and onto.
Find the properties of the empty relation $\emptyset \subset S \times S$ and the universal relation $U = S \times S$. Assume that $S$ is a nonempty domain.

(a) $\emptyset$ is (AR), (S), (AS), (T); if $S = \emptyset$ itself then $\emptyset$ is also (R).

(b) $U$ is (R), (S), (T); if $|S| \leq 1$ then also (AS)
3.1.10(a) Give examples of relations with specified properties. (AS), (T), ¬(R).

Examples over \( \mathbb{N}, \text{Pow}(\mathbb{N}) \)

- strict order of numbers \( x < y \)
- simple (weak) order, but with some pairs \( (x, x) \) removed from \( \mathcal{R} \)
- being a prime divisor
  \( (p, n) \in \mathcal{R} \) iff \( p \) is prime and \( p|n \)
  - not reflexive: \( (1, 1) \notin \mathcal{R}, (4, 4) \notin \mathcal{R}, (6, 6) \notin \mathcal{R} \)
  - transitivity is meaningful only for the pairs \( (p, p), (p, n), p|n \) for \( p \) prime
3.1.10(b) Give examples of relations with specified properties. (S), ¬(R), ¬(T).

Easiest examples - inequality

- $\mathcal{R} = \{(x, y) : x \neq y, \ x, y \in \mathbb{N}\}$
- $\mathcal{R} = \{(A, B) : A \neq B, \ A, B \subseteq S\}$
3.1.14 Which properties carry from individual relations to their union?

(a) $\mathcal{R}_1, \mathcal{R}_2 \in (R) \implies \mathcal{R}_1 \cup \mathcal{R}_2 \in (R)$

(b) $\mathcal{R}_1, \mathcal{R}_2 \in (S) \implies \mathcal{R}_1 \cup \mathcal{R}_2 \in (S)$

(c) $\mathcal{R}_1, \mathcal{R}_2 \in (T) \not\implies \mathcal{R}_1 \cup \mathcal{R}_2 \in (T)$

Eg. $S = \{a, b, c\}$, $a \mathcal{R}_1 b$, $b \mathcal{R}_2 c$

and no other relationships
Equivalence Relations and Partitions

Relation $\mathcal{R}$ is called an *equivalence* relation if it satisfies (R), (S), (T). Every equivalence $\mathcal{R}$ defines *equivalence classes* on its domain $S$.

The equivalence class $[s]$ (w.r.t. $\mathcal{R}$) of an element $s \in S$ is

$$[s] = \{ t \in S : t \mathcal{R} s \}$$

This notion is well defined only for $\mathcal{R}$ which is an equivalence relation. Collection of all equivalence classes $[S]_\mathcal{R} = \{ [s] : s \in S \}$ is a partition of $S$

$$S = \bigcup_{s \in S} [s]$$
Thus the equivalence classes are disjoint and jointly cover the entire domain. It means that every element belongs to one (and only one) equivalence class.

We call $s_1, s_2, \ldots$ representatives of (different) equivalence classes. For $s, t \in S$ either $[s] = [t]$, when $s \mathcal{R} t$, or $[s] \cap [t] = \emptyset$, when $s \not\mathcal{R} t$. We commonly write $s \sim_{\mathcal{R}} t$ when $s, t$ are in the same equivalence class.

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \cup \ldots \cup S_k$, then we specify $s \sim t$ exactly when $s$ and $t$ belong to the same $S_i$. 
If the relation $\sim$ is an equivalence on $S$ and $[S]$ the corresponding partition, then

$$\nu : S \rightarrow [S], \quad \nu : s \mapsto [s] = \{ x \in S : x \sim s \}$$

is called the *natural* map. It is always surjective.

**Question**

*When is $\nu$ also 1–1?*
If the relation $\sim$ is an equivalence on $S$ and $[S]$ the corresponding partition, then

$$
\nu : S \rightarrow [S], \quad \nu : s \mapsto [s] = \{ x \in S : x \sim s \}
$$

is called the natural map. It is always onto.

**Question**

*When is $\nu$ also 1–1?*

**Answer**

*When $\sim$ is the identity on $S$.***
A function $f : S \longrightarrow T$ defines an equivalence relation on $S$ by

$s_1 \sim s_2$ iff $f(s_1) = f(s_2)$

These sets $f^{-1}(t), \ t \in T$ that are nonempty form the corresponding partition

$$S = \bigcup_{t \in T} f^{-1}(t)$$

**Question**

*When are all $f^{-1}(t) \neq \emptyset$?*
A function \( f : S \rightarrow T \) defines an equivalence relation on \( S \) by

\[
s_1 \sim s_2 \quad \text{iff} \quad f(s_1) = f(s_2)
\]

These sets \( f^{-1}(t), \ t \in T \) that are nonempty form the corresponding partition

\[
S = \bigcup_{t \in T} f^{-1}(t)
\]

**Question**

*When are all \( f^{-1}(t) \neq \emptyset \)?*

**Answer**

*When \( f \) is onto.*
Example

Partition of $\mathbb{Z}$ into classes of numbers with the same remainder $(\text{mod } p)$; it is particularly important for $p$ prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime $p$; division has to be restricted when $p$ is not prime.

Standard notation:

$m = n \pmod{p}$ stands for: $m \mod p = n \mod p$

NB

$(\mathbb{Z}_p, +, \cdot, 0, 1)$ are fundamental algebraic structures known as rings. These structures are very important in coding theory and cryptography.
Example

3.6.6 (supp) Show that $m \sim n$ iff $m^2 = n^2 \pmod{5}$ is an equivalence on $S = \{1, \ldots, 7\}$. Find all the equivalence classes.

(a) It just means that $m = n \pmod{5}$ or $m = -n \pmod{5}$, e.g. $1 = -4 \pmod{5}$. This satisfies (R), (S), (T).

(b) We have
$[1] = \{1, 4, 6\}$
$[2] = \{2, 3, 7\}$
$[5] = \{5\}$
3.6.10 (supp)

\( \mathcal{R} \) is a relation on \( \mathbb{N} \times \mathbb{N} \), i.e. it is a subset of \( \mathbb{N}^4 \)
\( (m, n) \mathcal{R} (p, q) \) if \( m = p \pmod{3} \) or \( n = q \pmod{5} \).

(a) \( \mathcal{R} \in (R) \)?
Yes: \( (m, n) \sim (m, n) \) iff \( m = m \pmod{3} \) or \( n = n \pmod{5} \) iff true or true.

(b) \( \mathcal{R} \in (S) \)?
Yes: by symmetry of \( . \equiv . \pmod{n} \).

(c) \( \mathcal{R} \in (T) \)?
No — for arbitrary two pairs \( (m_1, n_1) \) and \( (m_2, n_2) \) one can create a chain \( (m_1, n_1) \mathcal{R} (m_2, n_1) \) and \( (m_2, n_1) \mathcal{R} (m_2, n_2) \), but not all pairs are related.
Order Relations

**Total order** \( \leq \) on \( S \)

(R) \( x \leq x \) for all \( x \in S \)

(AS) \( x \leq y, y \leq x \Rightarrow x = y \)

(T) \( x \leq y, y \leq z \Rightarrow x \leq z \)

(L) *Linearity* — any two elements are comparable:

for all \( x, y \) either \( x \leq y \) or \( y \leq x \) (and both if \( x = y \))
On a finite set all total orders are “isomorphic”

\[ x_1 \leq x_2 \leq \cdots \leq x_n \]

On an infinite set there is quite a variety of possibilities.

**Examples**
- discrete with a least element, e.g. \( \mathbb{N} = \{0, 1, 2, \ldots\} \)
- discrete without a least element, e.g. \( \mathbb{Z} = \{\ldots, 0, 1, 2, \ldots\} \)
- various dense/locally dense orders
  - rational numbers \( \mathbb{Q} : \forall p, q \in \mathbb{Q} (p < q \Rightarrow \exists r \in \mathbb{Q} (p < r < q)) \)
  - \( S = [a, b] \) — both least and greatest elements
  - \( S = (a, b] \) — no least element
  - \( S = [a, b) \) — no greatest element
  - other \( [0, 1] \cup [2, 3] \cup [4, 5] \cup \ldots \)
A **partial order** \( \preceq \) on \( S \) satisfies (R), (AS), (T); need not be (L)

We call \((S, \preceq)\) a **poset** — partially ordered set

Every finite poset can be represented as a **Hasse diagram**, where a line is drawn *upward* from \( x \) to \( y \) if \( x \prec y \) and there is no \( z \) such that \( x \prec z \prec y \)

11.1.1(a) Hasse diagram for positive divisors of 24

\[
p \preceq q \text{ if, and only if, } p \mid q
\]
Ordering Concepts

- Minimal and maximal elements (they always exist in every finite poset)
- Minimum and maximum — unique minimal and maximal element
- lub (least upper bound) and glb (greatest lower bound) of a subset $A \subseteq S$ of elements
  - lub$(A)$ — smallest element $x \in S$ s.t. $x \succeq a$ for all $a \in A$
  - glb$(A)$ — greatest element $x \in S$ s.t. $x \preceq a$ for all $a \in A$
- Lattice — a poset where lub and glb exist for every pair of elements
  (by induction, they then exist for every finite subset of elements)
Examples

• \( \text{Pow}(\{a, b, c\}) \) with the order \( \subseteq \)

  \( \emptyset \) is minimum; \( \{a, b, c\} \) is maximum

• \( 11.1.4 \)

  \( \text{Pow}(\{a, b, c\}) \setminus \{\{a, b, c\}\} \) (proper subsets of \( \{a, b, c\} \))
  Each two-element subset \( \{a, b\}, \{a, c\}, \{b, c\} \) is maximal.

  • But there is no maximum

• \( \{1, 2, 3, 4, 6, 8, 12, 24\} \) partially ordered by divisibility is a lattice

  • e.g. \( \text{lub}(\{4, 6\}) = 12; \text{glb}(\{4, 6\}) = 2 \)

• \( \{1, 2, 3\} \) partially ordered by divisibility is not a lattice

  • \( \{2, 3\} \) has no lub

• \( \{2, 3, 6\} \) partially ordered by divisibility is not a lattice

  • \( \{2, 3\} \) has no glb

• \( \{1, 2, 3, 12, 18, 36\} \) partially ordered by divisibility is not a lattice

  • \( \{2, 3\} \) has no lub (12, 18 are minimal upper bounds)
NB

An infinite lattice need not have a lub (or no glb) for an arbitrary infinite subset of its elements, in particular no such bound may exist for all its elements.

Examples

- $\mathbb{Z}$ — neither lub nor glb;
- $\mathbb{F}(\mathbb{N})$ — all finite subsets, has no arbitrary lub property; glb exists, it is the intersection, hence always finite;
- $\mathbb{I}(\mathbb{N})$ — all infinite subsets, may not have an arbitrary glb; lub exists, it is the union, which is always infinite.
Consider poset \((\mathbb{R}, \leq)\)
(a) Is this a lattice?
(b) Give an example of a non-empty subset of \(\mathbb{R}\) that has no upper bound.
(c) Find \(\text{lub}(\{ x \in \mathbb{R} : x < 73 \})\)
(d) Find \(\text{lub}(\{ x \in \mathbb{R} : x \leq 73 \})\)
(e) Find \(\text{lub}(\{ x : x^2 < 73 \})\)
(f) Find \(\text{glb}(\{ x : x^2 < 73 \})\)
Example

11.1.5 Consider poset \((\mathbb{R}, \leq)\)

(a) It is a lattice.

(b) subset with no upper bound: \(\mathbb{R}_{>0} = \{ r \in \mathbb{R} : r > 0 \}\)

(c) and (d) \(\text{lub}\{ x : x < 73 \} = \text{lub}\{ x : x \leq 73 \} = 73\)

(e) \(\text{lub}\{ x : x^2 < 73 \} = \sqrt{73}\)

(f) \(\text{glb}\{ x : x^2 < 73 \} = -\sqrt{73}\)
**Example**

11.1.13 $\mathbf{F}(\mathbb{N})$ — collection of all *finite* subsets of $\mathbb{N}$

(a) Does it have a maximal element?
(b) Does it have a minimal element?
(c) Given $A, B \in \mathbf{F}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbf{F}(\mathbb{N})$?
(d) Given $A, B \in \mathbf{F}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbf{F}(\mathbb{N})$?
(e) Is $\mathbf{F}(\mathbb{N})$ a lattice?
Example

11.1.13 \( F(\mathbb{N}) \) — collection of all finite subsets of \( \mathbb{N} \)
(a) No maximal elements
(b) \( \emptyset \) is the minimum
(c) \( \text{lub}(A, B) = A \cup B \)
(d) \( \text{glb}(A, B) = A \cap B \)
(e) \( F(\mathbb{N}) \) is a lattice — is has finite union and intersection properties.
11.1.14 $\mathbb{I}(\mathbb{N}) = \text{Pow}(\mathbb{N}) \setminus \mathbb{F}(\mathbb{N})$ — collection of all \textit{infinite} subsets of $\mathbb{N}$

(a) Does it have a maximal element?
(b) Does it have a minimal element?
(c) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbb{I}(\mathbb{N})$?
(d) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbb{I}(\mathbb{N})$?
(e) Is $\mathbb{I}(\mathbb{N})$ a lattice?
Example

11.1.14 \( \mathbb{I}(\mathbb{N}) = \text{Pow}(\mathbb{N}) \setminus \text{F}(\mathbb{N}) \) — collection of all infinite subsets of \( \mathbb{N} \)

(a) \( \mathbb{N} \) is the maximum

(b) No minimal elements (\( \emptyset \) is not in \( \mathbb{I}(\mathbb{N}) \))

(c) \( \text{lub}(A, B) = A \cup B \)

(d) \( \text{glb}(A, B) = A \cap B \) if it exists; it does not exist when \( A \cap B \) is finite, eg. when empty.

(e) \( \mathbb{I}(\mathbb{N}) \) is not a lattice — it has finite union but not finite intersection property; eg. sets \( 2\mathbb{N} \) and \( 2\mathbb{N} + 1 \) have the empty intersection.
Well-Ordered Sets

Well-ordered set: every subset has a least element.

**NB**

*The greatest element is not required.*

**Examples**

- $\mathbb{N} = \{0, 1, \ldots\}$
- $\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 \cup \ldots$, where each $\mathbb{N}_i \cong \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \ldots$

**NB**

*Well-order sets are an important mathematical tool to prove termination of programs.*
For a poset \((S, \preceq)\) any linear order \(\leq\) that is consistent with \(\preceq\) is called topological sort. Consistency means that \(a \preceq b \Rightarrow a \leq b\).

Consider

\[
\begin{array}{cccc}
  a & c & d \\
  b & e & f \\
\end{array}
\]

Various possible topological sortings

The following all are topological sorts:
\[
\begin{align*}
  a & \leq b \leq e \leq c \leq f \leq d \\
  a & \leq e \leq b \leq f \leq c \leq d \\
  \ldots & \\
  a & \leq e \leq f \leq b \leq c \leq d
\end{align*}
\]
Combining Orders

**Product order** — can combine any partial orders. In general, it is only a *partial order*, even if combining total orders.

For \( s, s' \in S \) and \( t, t' \in T \) define

\[
(s, t) \preceq (s', t') \quad \text{if} \quad s \preceq s' \quad \text{and} \quad t \preceq t'
\]
They are, effectively, *total* orders on the *product* of ordered sets.

- **Lexicographic order** — defined on all of $\Sigma^*$. It extends a total order already assumed to exist on $\Sigma$.
- **Lenlex** — the order on (potentially) the entire $\Sigma^*$, where the elements are ordered first by length. $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of $\Sigma^*$.
- **Filing order** — lexicographic order confined to the strings of the same length. It defines total orders on $\Sigma^i$, separately for each $i$. 
Example

11.2.5 Let $B = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $B^*$ in the
(a) Lexicographic order
$000, 0010, 010, 10, 1000, 101, 11$
(b) Lenlex order
$10, 11, 000, 010, 101, 0010, 1000$

11.2.8 When are the lexicographic order and lenlex on $\Sigma^*$ the same?
Only when $|\Sigma| = 1$. 
11.6.6 True or false?
(a) If a set $\Sigma$ is totally ordered, then the corresponding lexicographic partial order on $\Sigma^*$ also must be totally ordered.
(b) If a set $\Sigma$ is totally ordered, then the corresponding lenlex order on $\Sigma^*$ also must be totally ordered.
(c) Every finite partially ordered set has a Hasse diagram.
(d) Every finite partially ordered set has a topological sorting.
(e) Every finite partially ordered set has a smallest element.
(f) Every finite totally ordered set has a largest element.
(g) An infinite partially ordered set cannot have a largest element.
Supplementary Exercises

11.6.6
(a) and (b) – True; this is the idea behind various lex-sorts
(c) Yes.
(d) Yes.
(e) False – consider a two-element set with the identity as p.o.
(f) True – due to the finiteness
(g) False, eg. $\mathbb{Z}_{<0}$
Matrices

An $m \times n$ matrix is a rectangular array with $m$ horizontal rows and $n$ vertical columns.

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

NB

Matrices are important objects in Computer Science, e.g. for

- optimisation
- graphics and computer vision
- cryptography
- information retrieval and web search
- machine learning
Basic Matrix Operations

The **transpose** $A^T$ of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix whose entry in the $i$th row and $j$th column is $a_{ji}$.

**Example**

$$A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \quad \quad A^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

**NB**

A matrix $M$ is called symmetric if $M^T = M$
The sum of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix whose entry in the $i$th row and $j$th column is $a_{ij} + b_{ij}$.

**Example**

\[
A = \begin{bmatrix}
2 & -1 & 0 & 4 \\
3 & 2 & -1 & 2 \\
4 & 0 & 1 & 3 \\
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 0 & 5 & 3 \\
2 & 3 & -2 & 1 \\
4 & -2 & 0 & 2 \\
\end{bmatrix}
\]

\[
A + B = \begin{bmatrix}
3 & -1 & 5 & 7 \\
5 & 5 & -3 & 3 \\
8 & -2 & 1 & 5 \\
\end{bmatrix}
\]

**Fact**

$A + B = B + A$ and $(A + B) + C = A + (B + C)$
Given $m \times n$ matrix $A = [a_{ij}]$ and $c \in \mathbb{R}$, the scalar product $cA$ is the $m \times n$ matrix whose entry in the $i$th row and $j$th column is $c \cdot a_{ij}$.

**Example**

$$A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \quad 2A = \begin{bmatrix} 4 & -2 & 0 & 8 \\ 6 & 4 & -2 & 4 \\ 8 & 0 & 2 & 6 \end{bmatrix}$$
The **product** of an \( m \times n \) matrix \( \mathbf{A} = [a_{ij}] \) and an \( n \times p \) matrix \( \mathbf{B} = [b_{jk}] \) is the \( m \times p \) matrix \( \mathbf{C} = [c_{ik}] \) defined by

\[
c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq k \leq p
\]

**Example**

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
= \begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\
a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}
\end{bmatrix}
\]

**NB**

*The rows of \( \mathbf{A} \) must have the same number of entries as the columns of \( \mathbf{B} \).*

*The product of a \( 1 \times n \) matrix and an \( n \times 1 \) matrix is usually called the **inner product** of two \( n \)-dimensional vectors.*
Example

Consider
\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \]

Calculate \( AB \), \( BA \)
\[ AB = \begin{bmatrix} -10 & 5 \\ -20 & 10 \end{bmatrix} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

NB
In general, \( A \cdot B \neq B \cdot A \)
Example: Computer Graphics

Rotating an object w.r.t. the $x$ axis by degree $\alpha$:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha \\
\end{bmatrix}
\cdot
\begin{bmatrix}
5 & 5 & 7 & 7 & 5 & 7 & 5 & 7 & 7 \\
1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\
9 & 7 & 7 & 9 & 7 & 7 & 9 & 9 & 9 \\
\end{bmatrix}
\]
Summary

- Properties of functions: onto, 1-1; $f^{-1}$, $f^\leftarrow$
- Properties of binary relations: (R), (AR); (S), (AS); (T)
- Matrix operations: transposition, sum, scalar product, product
- Equivalence relations $\sim$, equivalence classes $[S]$, example $\mathbb{Z}_p$
- Ordering concepts: total, partial, lub, glb, lattice, topological sort
- Orderings: product, lexicographic, lenlex, filing