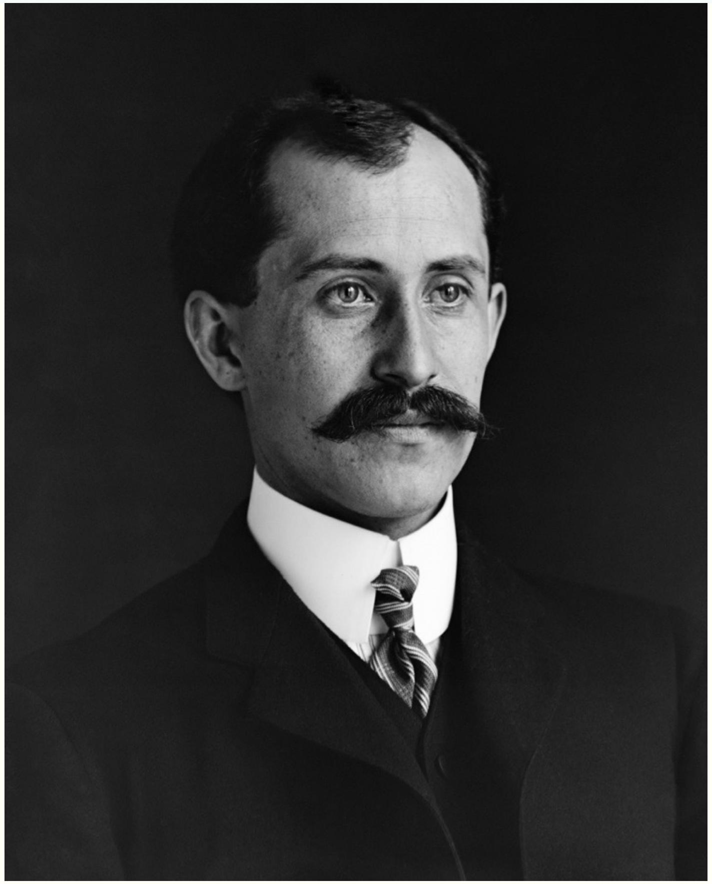
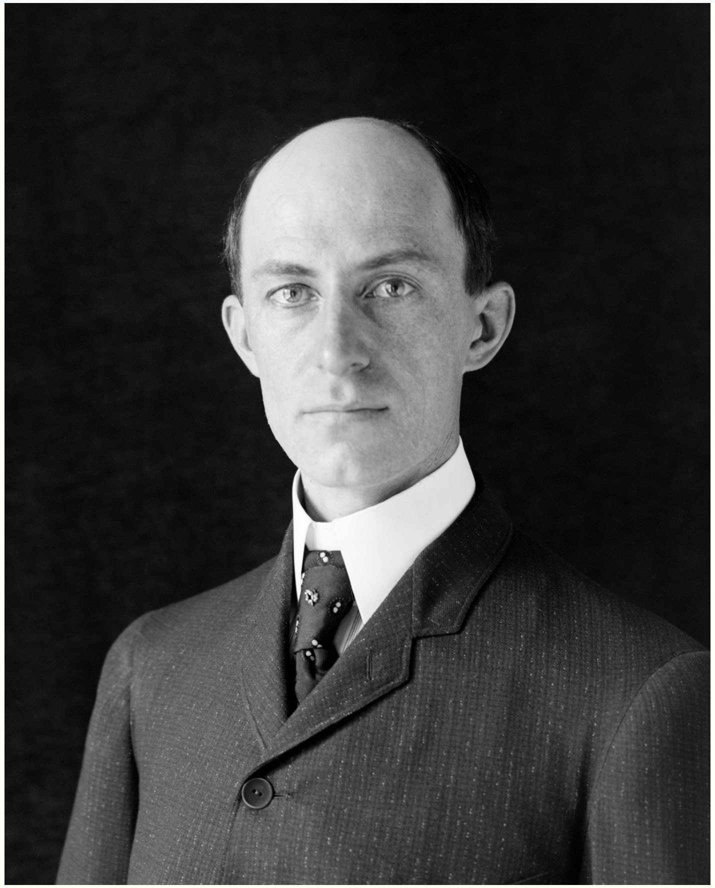
ENGG1811 Computing for Engineers

Week 8A: Simulation

Wright brothers





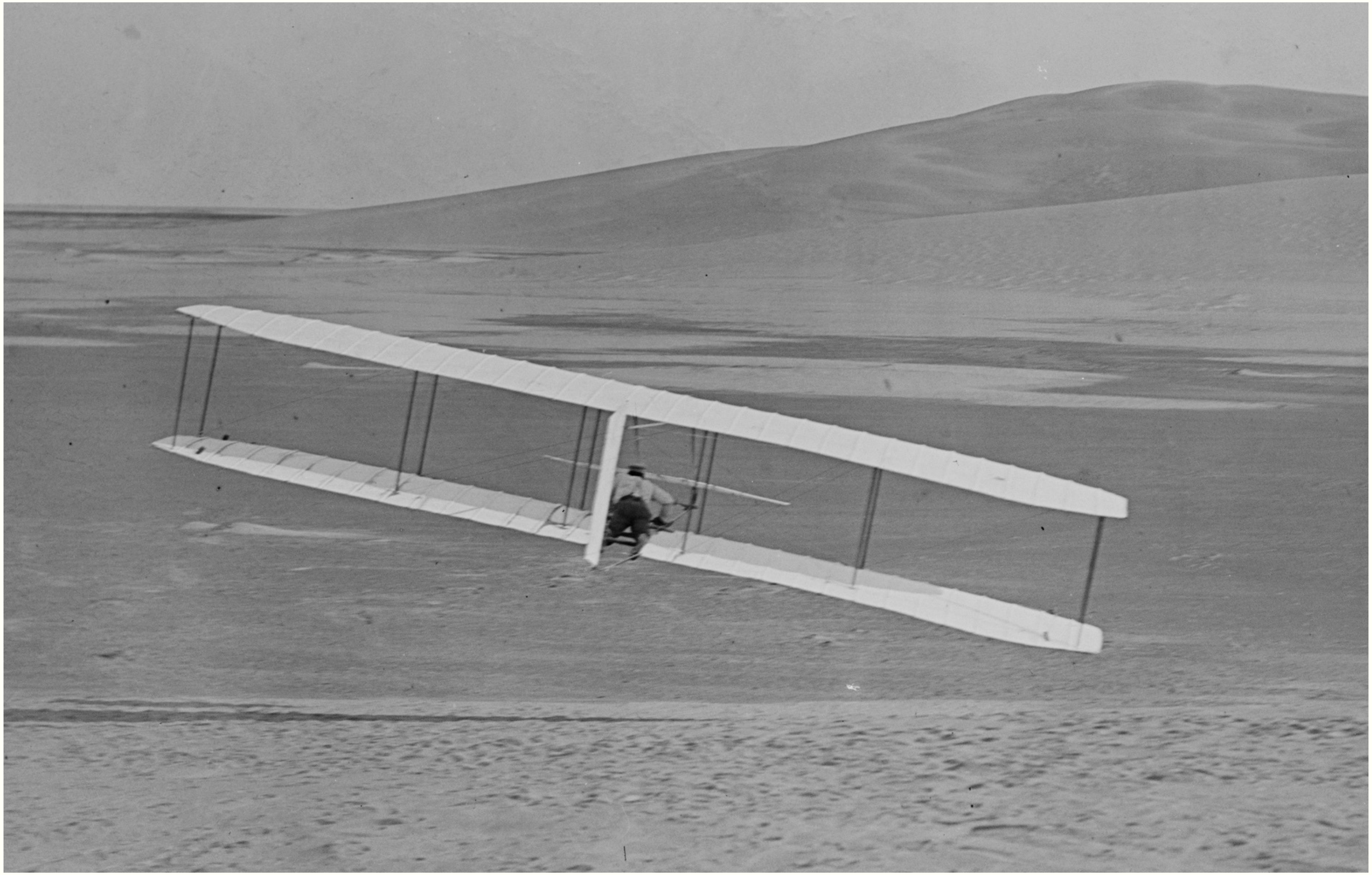
Invented and built the world's first powered airplane

Pictures from http://en.wikipedia.org/wiki/Wright_brothers

Crumpled gilder, Oct 1900

http://www.theatlantic.com/photo/2014/08/first-flight-with-the-wright-brothers/100796/

Glider (i.e. no power) (1902)



First powered flight (17 Dec 1903)
(Added: Propeller, engine)

Classical engineering design iteration

1. Design
   - This step may use calculations, physical laws, chemistry or biology, experimental data, intuition and guesses
2. Build
3. Test
4. If it doesn’t work, go back to design (Step 1).

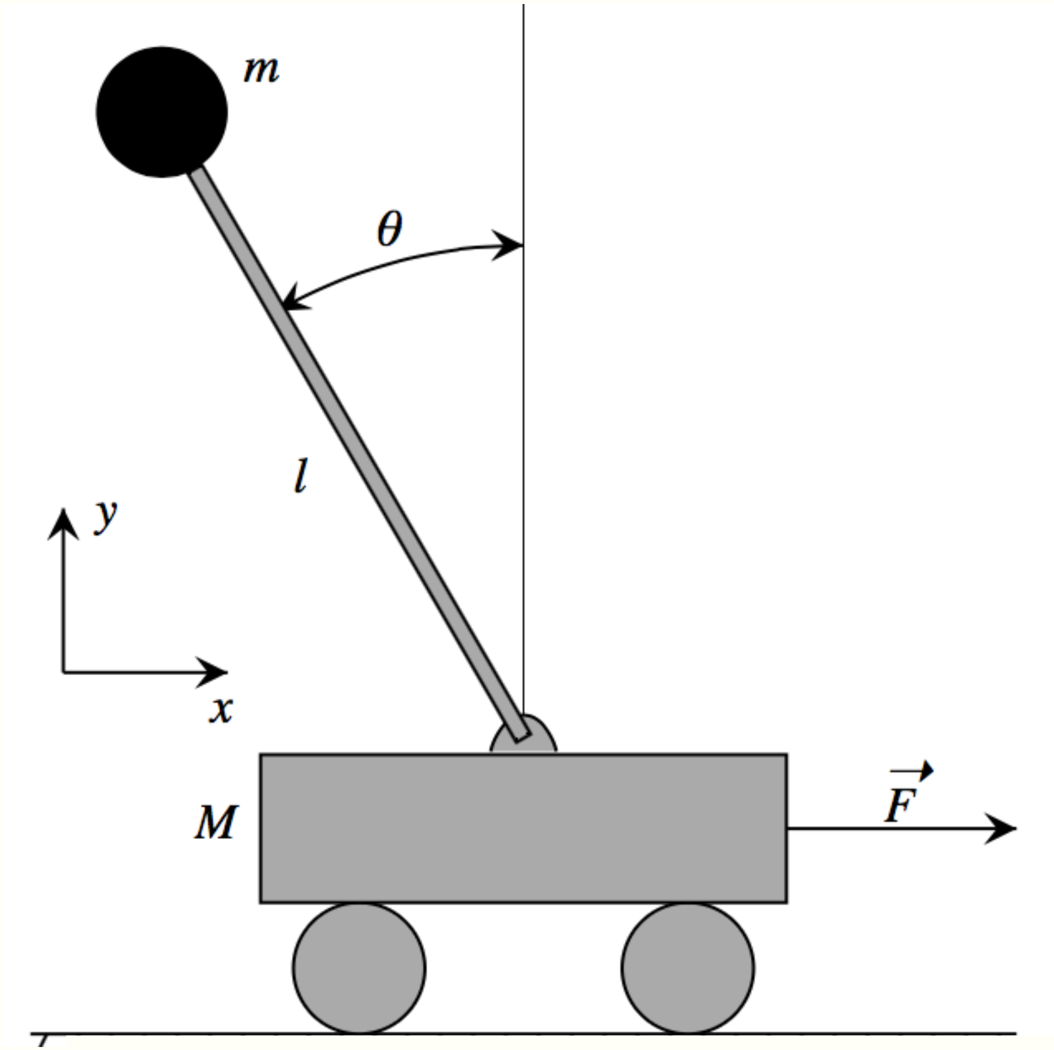
Engineering design iteration – with computers

1. Design on computers
   a) Derive **mathematical model** of the design
   b) Perform calculations, **simulations** or optimisation to understand or improve design
   c) Reject designs with poor performance. If none of the designs is good, go back to (a) for a new design or (b) to try to optimise the design.
   d) Choose one or more candidates for prototyping or building the actual design
2. Build
3. Test
4. If it doesn't work, go back to design (Step 1).

Mathematical model can be derived from science (maths, physics, chemistry, biophysics) or data

Design challenge: Balancing an inverted pendulum

- Can you balance a stick on your finger tip/palm

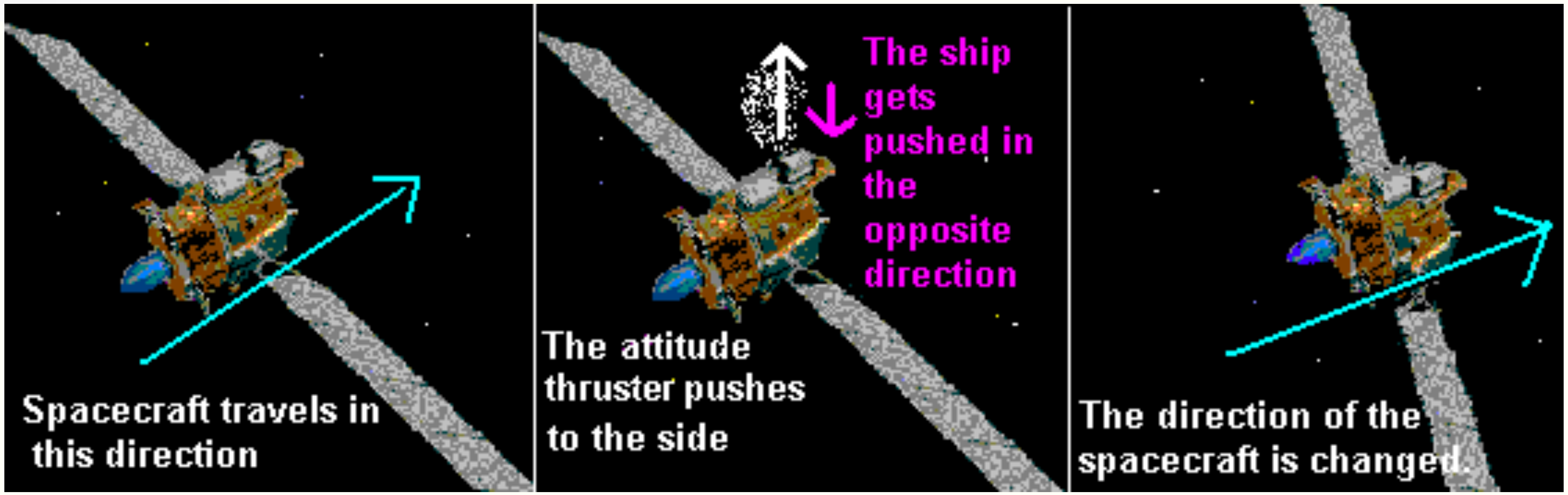


- An inverted pendulum is sitting on a cart.
- The aim of the design is to balance the inverted pendulum by applying an appropriate force on the cart.

Picture: http://en.wikipedia.org/wiki/Inverted_pendulum

Applications of inverted pendulum

- Segway
- Rocket/spaceship attitude control
  - i.e., orientation control



Picture http://www.segway.com/

http://www.qrg.northwestern.edu/projects/vss/docs/propulsion/2-what-is-attitude-control.html

This week

- Simulation
- Python components
  - Some new numpy functions
- Mathematical / physics / chemistry concepts
  - Mathematical modelling
  - Numerical approximation of derivatives
  - Ordinary differential equations

More on numpy

- Before looking at simulation, we will first go through a number of numpy functions which are related to our discussion this week
- The numpy functions are:
  - arange(), linspace(), zeros(), ones(), zeros_like()
- The file is in numpy_ex.py

Notation in the lecture notes

- We will be using both mathematical variables and Python variables in this lecture
- We may say the position of an object at time t is $x(t)$
  - For example, x(0.3) = 5 says that the object is at the position 5 at time 0.3
- We may store the position of the object in a numpy array

Notation

Real number with interpretation of time

- Mathematical variable : x(0.4)

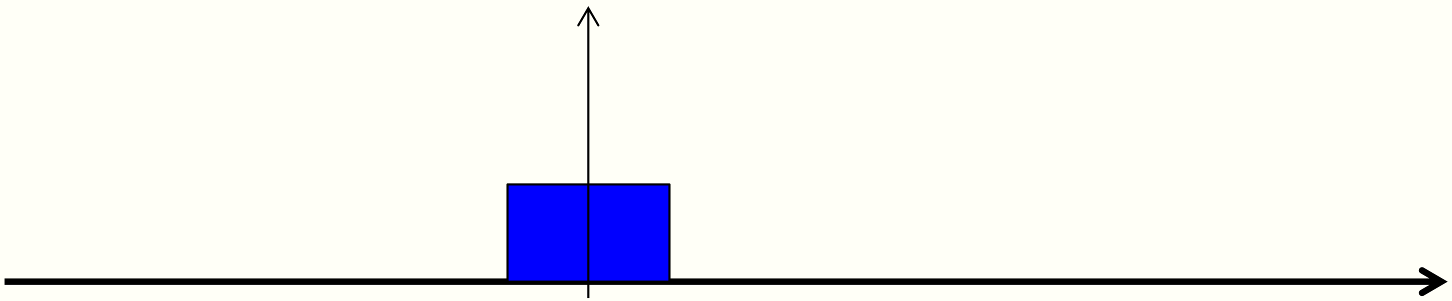
- Numpy array: position[5]

Zero or positive integers only
An index to an array
We assume we don't use –ve indexing here

- A simple way to remember:
  - Mathematical variables: ()
  - Numpy array: []

Simulation on paper – the setup

- An object is constrained to move along a straight line
- Time starts at 0 unit. The initial position of the object is $x(0) = 1$
- The velocity $v(t)$ at time t is:
  - $v(t) = 2$ if $0 \le t < 0.4$
  - $v(t) = -5$ if $0.4 \le t$
- Determine the position of the object at t = 0.1,0.2, …, 0.6



At position 1 at time 0

Calculating positions on paper

- Given:
  - Initial position x(0) = 1
  - Velocity in time interval [0,0.1] is 2
- Aim: Find the position at time 0.1 = x(0.1)

- $x(0.1) = x(0) + 2 * 0.1 = 1.2$

- How about position at time 0.2 = x(0.2)
  - Velocity in time interval [0.1,0.2] is 2

- $x(0.2) = x(0.1) + 2 * 0.1 = 1.4$

Quiz: Position at time 0.3

- Given
  - $x(0.2) = 1.4$
  - Velocity in time interval [0.2,0.3] is 2
- What is the position at time 0.3?
  - Equivalently: What is $x(0.3)$?

Python variable for time instances

- Our aim is to compute the position of the object at time instances 0, 0.1, 0.2, … , 0.6

- We want to create a numpy array whose elements are:
  - [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6]
  - Python variable name for this numpy array: time_array
  - We can generate this array by using either arange() or linspace()

Python variable for positions

- time_array = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6]
- pos_array = [1, 1.2, 1.4, …]
- pos_array and time_array have the same shape
- Note:
  - pos_array[0] = position at time 0 = position at time time_array[0]
  - pos_array[1] = position at time 0.1 = position at time time_array[1]
- Generally:
  - pos_array[k] = position at time 0.1*k = position at time time_array[k]

Mapping on paper simulation to Python code

- On paper calculations:
  - x(0) = position at time 0
  - x(0.1) = position at time 0.1
  - The calculation is:

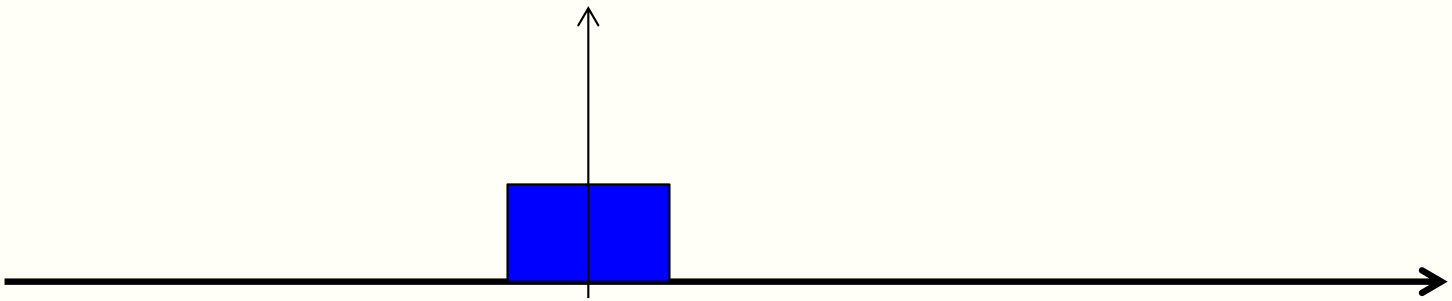
$$x(0.1) = x(0) + 2 * 0.1 = 1.2$$

- Python code:
  - pos_array[0] stores the position at time 0
  - pos_array[1] stores the position at time 0.1
  - Python code is:

```
pos_array[1] = pos_array[0] + 2 * 0.1
```

Simulation on paper – the setup (repeat)

- An object is constrained to move along a straight line
- Time starts at 0 unit. The initial position of the object is $x(0) = 1$
- The velocity $v(t)$ at time t is:
  - $v(t) = 2$ if $0 \le t < 0.4$
  - $v(t) = -5$ if $0.4 \le t$
- Determine the position of the object for $t = 0.1, 0.2, \ldots, 0.6$



At position 1 at time 0

Simulation on paper

Index k	time_array[k]	Velocity at the current time	Position pos_array[k+1] Note: pos_array[0] = 1
0	0.0	2	pos_array[1] = pos_array[0] + 2 * 0.1 = 1.2
1	0.1	2	pos_array[2] = pos_array[1] + 2 * 0.1 = 1.4
2	0.2	2	pos_array[3] = pos_array[2] + 2 * 0.1 = 1.6
3	0.3	2	
4	0.4	-5	
5	0.5	-5	
6	0.6		

Let us complete the Python implementation in simulate_1d_prelim.m

simulate_1d.py (simulation loop only)

```python
for k in range(len(time_array)-1):
    #  Current time
    time_now = time_array[k]

    #  Velocity at the current time
    if time_now < TIME_LIMIT_1:
        velocity_now = VELOCITY_1
    else:
        velocity_now = VELOCITY_2

    # Compute pos_array[k+1]
    pos_array[k+1] = pos_array[k] + velocity_now * dt
```

Week 3's in-lecture project (1)

- We just wrote a simulation program to determine the position of a block over time, you used a different method to determine the speed of an object over time in Week 3.

- Speed of an object in freefall

$$v(t) = \frac{gm}{d}\left(1 - e^{-\frac{d}{m}t}\right)$$

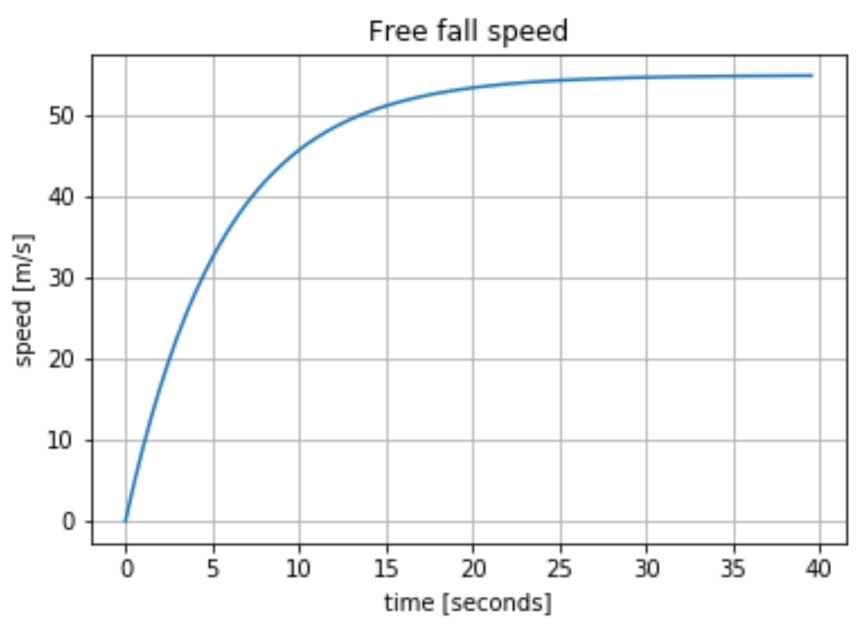
- You created a list of time instants

[0, 0.5, 1, 1.5, 2, 2.5,   39.5, 40]

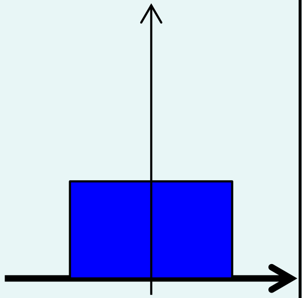
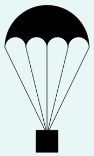
Week 3's in-lecture project (2)

- You use for-loops to create a list of speeds
  - Time is 0. Use the Speed formula. Speed = 0.
  - Time is 0.5. Use the speed formula. Speed = 4.692400935
  - Time is 1. Use the speed formula. Speed = 8.98399681455

  - Time is 40. Use the speed formula. Speed = 54.8885179036



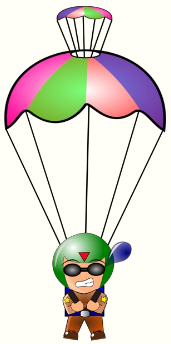
Contrasting two methods to do simulation

Methods / Problems	By increment	By Formula
	pos_array[k+1] = pos_array[k] + velocity * dt	
		$$v(t) = \frac{gm}{d}\left(1 - e^{-\frac{d}{m}t}\right)$$

Simulation by formula

Jump at time 0
Speed = 0

freefall



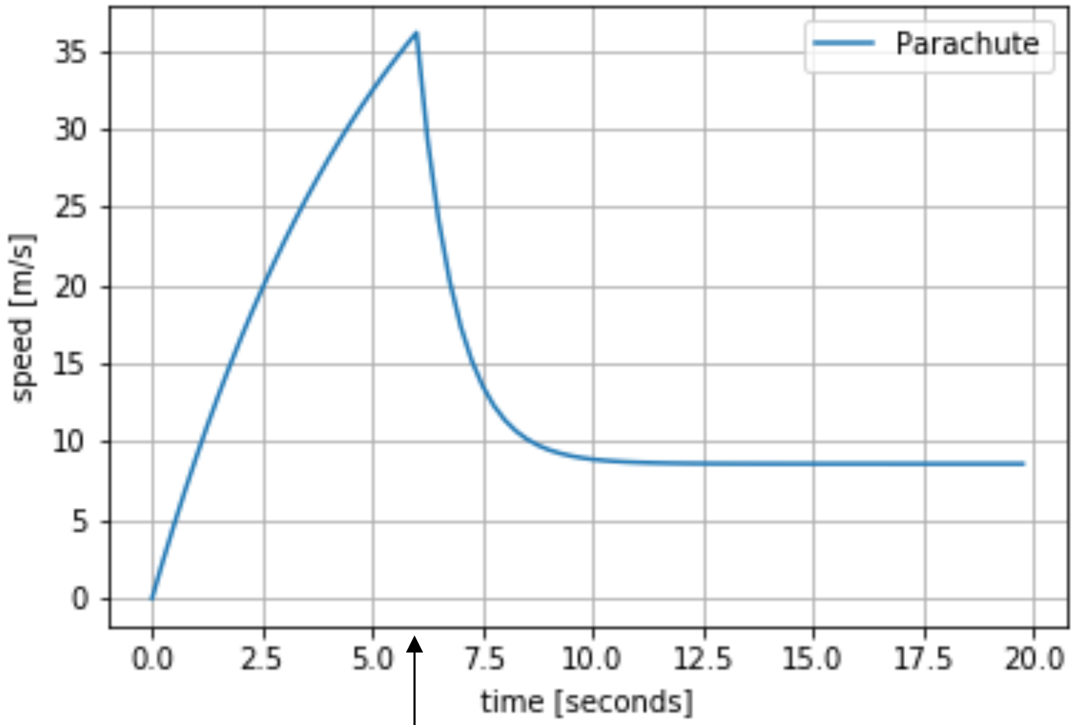
Parachute deployed after 6 seconds

Retarded fall

- A parachutist jumps from the plane, we want to calculate their speed over time and plot the speed profile

The parachutist's speed profile

- We will need two formulas
  - One before the parachute is deployed: freefall
  - One after the parachute is deployed



Time at which the parachute is deployed

Formula #1: Before the parachute is deployed

- Notation:
  - m is the mass of the parachutist
  - g is acceleration due to gravity ($\mathrm{m\,s^{-2}}$)
  - c_{air} is the drag coefficient in air (in $\mathrm{kg\,s^{-1}}$)
  - t_c is the time the parachute is deployed
- The speed of the parachutist before the parachute is deployed is given by the formula:

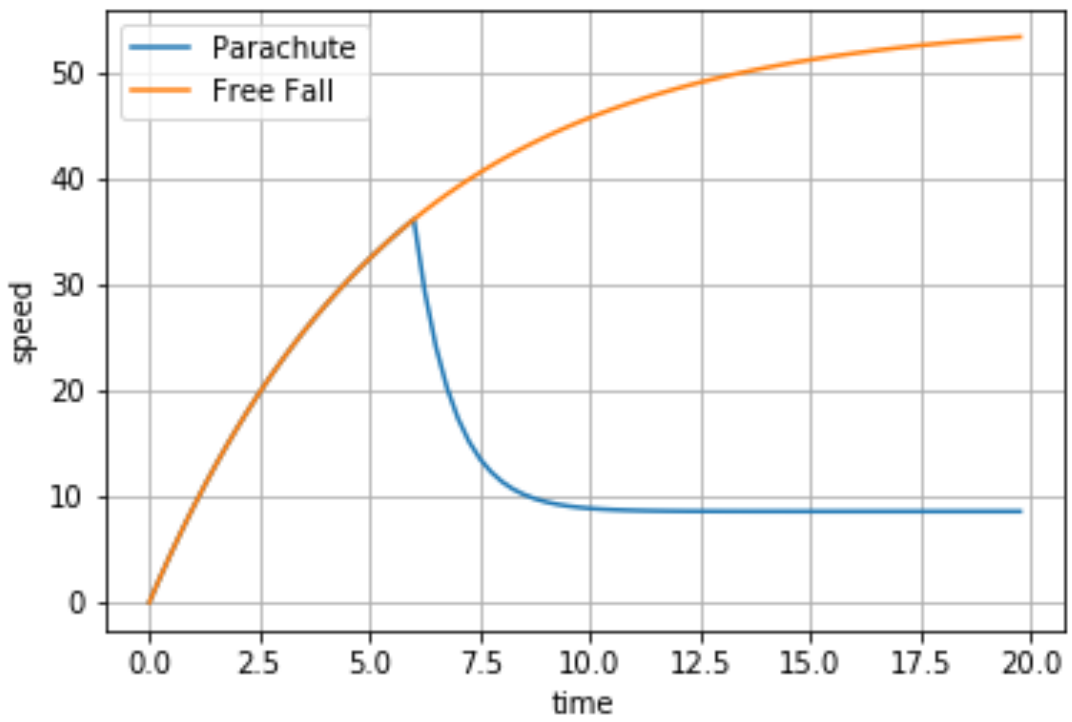
Free fall part

If $t < t_c$

$$v(t) = \frac{gm}{c_{\mathrm{air}}}\left(1 - e^{-\frac{c_{\mathrm{air}}}{m}t}\right)$$

Some intuition

$$v(t) = \frac{gm}{c_{\text{air}}}\left(1 - e^{-\frac{c_{\text{air}}}{m}t}\right)$$



The exponential factor decays in magnitude, so the speed asymptotically approaches $g\, m/\, c_{air}$

For a free-falling 70kg parachutist with c_{air} = 12.5, this **terminal speed** is ~55 m s^{-2} (200km/hr)

Formula #2: After the parachute is deployed

t_c = the time at which the parachute is deployed

If $t \geq t_c$

$$v_{p0} = \frac{gm}{c_{\text{air}}}\left(1 - e^{-\frac{c_{\text{air}}}{m}t_c}\right)$$

Speed at the moment parachute is deployed

$$v(t) = v_{p0}e^{-\frac{c_{dp}}{m}(t-t_c)} + \frac{gm}{c_{dp}}\left(1 - e^{-\frac{c_{dp}}{m}(t-t_c)}\right)$$

A larger drag coefficient c_{dp}, i.e., $c_{dp} > c_{air}$

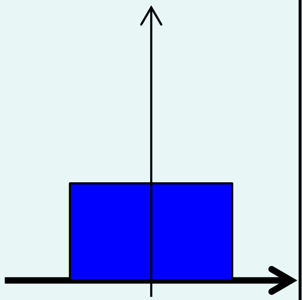
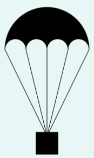
Parachutist simulation

- We've written a function that, given all parameters, calculates the speed at any time t
- The algorithm, expressed in *pseudocode*, is

```
for t in time_array
    if t < tc   # still in free-fall
        Calculate the speed using the freefall formula
    else # parachute has been deployed
        Calculate velocity at time of deployment
        Calculate velocity using the parachute formula
```

Code in para_speed_by_formula.py and para_formula_lib.py

Comparing object moving in 1D and parachutist

Methods / Problems	By increment	By Formula
	pos_array[k+1] = pos_array[k] + velocity * dt	• If $0 \le t \le 0.4$, $x(t) = 1+2t$ • If $0.4 \le t$, $x(t) = 1.8 - 5\,(t - 0.4)$
	?	$$v(t) = \frac{gm}{d}\left(1 - e^{-\frac{d}{m}t}\right)$$ **plus others.**

An inconvenient truth

- Solving problems by deriving a formula
  - Mathematically elegant; exact solution
  - Formulas may provide insight
  - Convenient to use: simply perform substitution

Bad news!

Most advanced engineering problems do not have an **exact** solution in the form of a formula

Good news!

You can solve these problems **numerically** and **approximately** by **computers** and **programming**

Non-formula solution to the parachutist problem

- The velocity of the parachutist obeys the following ordinary differential equations (ODE)

$$\frac{dv(t)}{dt} = g - \frac{c(t)}{m} v(t)$$

- v(t) = speed at time t
- c(t) = drag coefficient at time t

- We will look at how you can solve this equation numerically and approximately.

Approximating derivatives

- From the definition of derivatives, we know

$$\frac{dv(t)}{dt} = \lim_{\Delta \to 0} \frac{v(t+\Delta) - v(t)}{\Delta}$$

- If Δ is small enough, then

$$\frac{dv(t)}{dt} \approx \frac{v(t+\Delta) - v(t)}{\Delta}$$

Code: approximate_derivative.py

Approximating derivatives – numerical illustration

- $f(x) = x^3$
- Derivative of $f(x) = f'(x) = 3x^2$
- At $x = 2$, $f'(2) = 12$

$$\lim_{\Delta \to 0} \frac{(2+\Delta)^3 - 2^3}{\Delta}$$

- Approximate method

$$\frac{(2+\Delta)^3 - 2^3}{\Delta}$$

- Let us try different values of Δ

Δ	
0.1000	12.61000000
0.0100	12.06010000
0.0010	12.00600100
0.0001	12.00060001

Solving ODE numerically (1)

1) Starting from the ODE

$$\frac{dv(t)}{dt} = g - \frac{c(t)}{m} v(t)$$

2) Replace the derivative by its approximation

$$\frac{dv(t)}{dt} \approx \frac{v(t+\Delta) - v(t)}{\Delta}$$

We obtain:

$$\frac{v(t+\Delta) - v(t)}{\Delta} \approx g - \frac{c(t)}{m} v(t)$$

Solving ODE numerically (2)

From last slide:

$$\frac{v(t+\Delta) - v(t)}{\Delta} \approx g - \frac{c(t)}{m} v(t)$$

3) Make v(t + Δ) the subject:

$$v(t+\Delta) \approx v(t) + (g - \frac{c(t)}{m} v(t))\Delta$$

Solving ODE numerically (3)

From last slide:

$$v(t+\Delta) \approx v(t) + \left(g - \frac{c(t)}{m}v(t)\right)\Delta$$

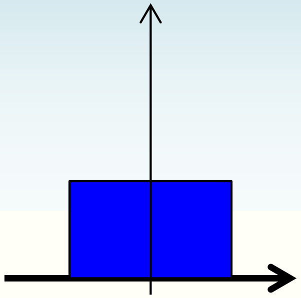
New speed **Current speed**

- For simulation, let us assume speed is stored in the array speed_array
- Identify

  v(t+Δ) with speed_array[k+1]

  v(t) with speed_array[k]

The analogy …



x(0.3) = x(0.2) + 2 * 0.1 = 1.6

Position after 0.1 time units

Position at time 0.2

Δ

---

pos_array[4] = pos_array[3] + 2 * 0.1 = 1.6

Next element in the array

Current element in the array

para_ODE_lib.py (simulation loop only)

```
for k in range(len(time_array)-1):
        # Current time
        time_now = time_array[k]

        # Determine the drag coefficient at time_now
        if time_now <= time_deploy:
            drag_coeff_now = drag_air
        else:
            drag_coeff_now = drag_para

        # Compute speed_array[k+1]
        speed_array[k+1] = speed_array[k] + \
            (g - drag_coeff_now * speed_array[k] / mass) * dt
```

Python code: approx ODE versus formula

- A Python function to solve the ODE numerically for the parachutist problem
  - Solution in the function: para_ODE_lib.py

- Note
  - Formula is exact
  - Numerical solution to ODE is an **approximation**

- Python script para_speed_by_ODE.py compares the formula against the approximate numerical solution

- We will vary the value of Δ, we expect
  - Small Δ, small difference between the two methods
  - And vice versa

Where did the ODE come from?

ODE we used. Multiply both sides by m.

$$\frac{dv(t)}{dt} = g - \frac{c(t)}{m} v(t)$$

Let us look at what this means.

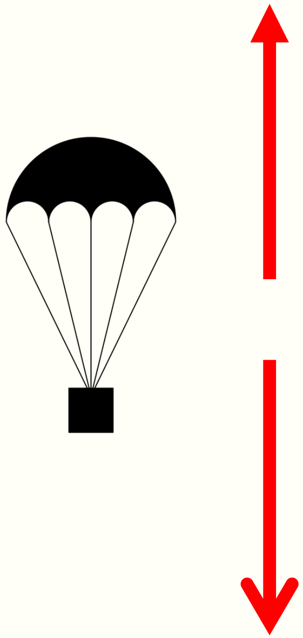
$$m \frac{dv(t)}{dt} = mg - c(t)v(t)$$

ODEs describe physical laws

$$m\frac{dv(t)}{dt} = mg - c(t)v(t)$$

mass x acceleration = Net downward force on the parachutist

What physical law is this?



c(t) v(t) = drag force

m g = gravitational pull

The big picture

- Physical law gives the ODE

$$m\frac{dv(t)}{dt} = mg - c(t)v(t)$$

- Computers and algorithms allow you to obtain numerical and approximate solution

- That's why you need to learn maths, physics, chemistry, your own disciplinary knowledge and COMPUTING!

Solving ODEs

- The method we use for solving ODE is known as **Euler's forward method**
- Meaning of forward and backward:

Forward: $$\frac{dv(t)}{dt} \approx \frac{v(t+\Delta) - v(t)}{\Delta}$$

Backward: $$\frac{dv(t)}{dt} \approx \frac{v(t) - v(t-\Delta)}{\Delta}$$

- Euler's forward method is simpler to explain but **not** the best. This is so you can focus on learning programming
- You will learn better methods in later years

The extended parachutist problem

- What if you want to determine the height of the parachutist too?
- Let h(t) = height of the parachutist at time t
- How can you compute h(t + Δ) from h(t)?

$$h(t+\Delta) \approx h(t) - v(t)\Delta$$

New height **Current height**

- You can formally derive this from the following ODE which says: derivative of height = downward speed

$$\frac{dh(t)}{dt} = -v(t)$$

Python implementation

- Essentially, two updates in the for loop

$$v(t+\Delta) \approx v(t) + (g - \frac{c(t)}{m}v(t))\Delta$$

$$h(t+\Delta) \approx h(t) - v(t)\Delta$$

- Python function: para_ODE_ext_lib.py
- Python script: para_speed_height_by_ODE.py
  - The script also illustrates how to plot with two different scales for the y-axis

para_ODE_ext_lib.py

Note: The changes, relative to para_ODE_lib.py is indicated in red.

```
def para_speed_height_ODE(time_array, mass, speed0,
               height0, drag_air, time_deploy, drag_para):

    height_array = np.zeros_like(time_array)

    height_array[0] = height0

    # simulation loop
    for k in range(len(time_array)-1):

        height_array[k+1] = height_array[k] - \
                            speed_array[k] * dt
```

Summary

- We have introduced the basics of simulation, which is a key tool in modern engineering and science
  - A formula solution is rare for modern day complex engineering problems
  - Numerical solution, approximation solution and simulation are important methods
- The basic method to do simulation is to set up an iteration step which can be obtained from ordinary differential equations