

Three Generalizations of the FOCUS Constraint

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Abstract

The FOCUS constraint expresses the notion that solutions are concentrated. In practice, this constraint suffers from the rigidity of its semantics. To tackle this issue, we propose three generalizations of the FOCUS constraint. We provide for each one a complete filtering algorithm as well as discussing decompositions.

1 Introduction

Many discrete optimization problems have constraints on the objective function. Being able to represent such constraints is fundamental to deal with many real world industrial problems. Constraint programming is a promising approach to express and filter such constraints. In particular, several constraints have been proposed for obtaining well-balanced solutions [Pesant and Régin, 2005; Schaus *et al.*, 2007; Petit and Régin, 2011]. Recently, the FOCUS constraint [Petit, 2012] was introduced to express the opposite notion. It captures the concept of concentrating the high values in a sequence of variables to a small number of intervals. We recall its definition. Throughout this paper, $X = [x_0, x_1, \dots, x_{n-1}]$ is a sequence of variables and $s_{i,j}$ is a sequence of indices of consecutive variables in X , such that $s_{i,j} = [i, i + 1, \dots, j]$, $0 \leq i \leq j < n$. We let $|E|$ be the size of a collection E .

Definition 1 ([Petit, 2012]). *Let y_c be a variable. Let k and len be two integers, $1 \leq len \leq |X|$. An instantiation of $X \cup \{y_c\}$ satisfies $\text{FOCUS}(X, y_c, len, k)$ iff there exists a set S_X of disjoint sequences of indices $s_{i,j}$ such that three conditions are all satisfied: (1) $|S_X| \leq y_c$ (2) $\forall x_l \in X, x_l > k \Leftrightarrow \exists s_{i,j} \in S_X$ such that $l \in s_{i,j}$ (3) $\forall s_{i,j} \in S_X, j - i + 1 \leq len$*

FOCUS can be used in various contexts including cumulative scheduling problems where some excesses of capacity can be tolerated to obtain a solution [Petit, 2012]. In a cumulative scheduling problem, we are scheduling activities, and each activity consumes a certain amount of some resource. The total quantity of the resource available is limited by a capacity. Excesses can be represented by variables [De Clercq *et al.*, 2011]. In practice, excesses might be tolerated by, for example, renting a new machine to produce more resource. Suppose the rental price decreases proportionally

to its duration: it is cheaper to rent a machine during a single interval than to make several rentals. On the other hand, rental intervals have generally a maximum possible duration. FOCUS can be set to concentrate (non null) excesses in a small number of intervals, each of length at most len .

Unfortunately, the usefulness of FOCUS is hindered by the rigidity of its semantics. For example, we might be able to rent a machine from Monday to Sunday but not use it on Friday. It is a pity to miss such a solution with a smaller number of rental intervals because FOCUS imposes that all the variables within each rental interval take a high value. Moreover, a solution with one rental interval of two days is better than a solution with a rental interval of four days. Unfortunately, FOCUS only considers the *number* of disjoint sequences, and does not consider their *length*.

We tackle those issues here by means of three generalizations of FOCUS. SPRINGYFOCUS tolerates within each sequence in $s_{i,j} \in S_X$ some values $v \leq k$. To keep the semantics of grouping high values, their number is limited in each $s_{i,j}$ by an integer argument. WEIGHTEDFOCUS adds a variable to count the length of sequences, equal to the number of variables taking a value $v > k$. The most generic one, $\text{WEIGHTEDSPRINGYFOCUS}$, combines the semantics of SPRINGYFOCUS and WEIGHTEDFOCUS . Propagation of constraints like these complementary to an objective function is well-known to be important [Petit and Poder, 2008; Schaus *et al.*, 2009]. We present and experiment with filtering algorithms and decompositions therefore for each constraint.

2 Springy FOCUS

In Definition 1, each sequence in S_X contains *exclusively* values $v > k$. In many practical cases, this property is too strong. Consider one simple instance of the problem in the introduction, in Figure 1, where one variable $x_i \in X$ is defined per point in time i (e.g., one day), to represent excesses of capacity. Initially, 4 activities are fixed and one activity a remains to be scheduled (drawing A), of duration 5 and that can start from day 1 to day 5. If $\text{FOCUS}(X, y_c = 1, 5, 0)$ is imposed then a must start at day 1 (solution B). We have one 5 day rental interval. Assume now that the new machine may not be used every day. Solution (C) gives one rental of 3 days instead of 5. Furthermore, if $len = 4$ the problem will have no solution using FOCUS, while this latter solution still exists in practice. This is paradoxical, as relaxing the condition

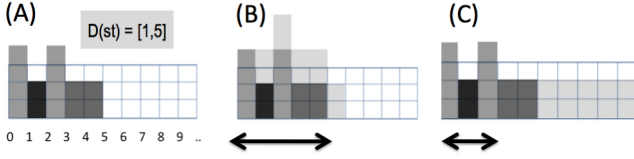


Figure 1: (A) Problem with 4 fixed activities and one activity of length 5 that can start from time 1 to 5. (B) Solution satisfying $\text{FOCUS}(X, [1, 1], 5, 0)$, with a new machine rented for 5 days. (C) Practical solution violating $\text{FOCUS}(X, [1, 1], 5, 0)$, with a new machine rented for 3 days but not used on the second day.

that sequences in the set S_X of Definition 1 take only values $v > k$ deteriorates the concentration power of the constraint. Therefore, we propose a soft relaxation of FOCUS, where at most h values less than k are tolerated within each sequence in S_X .

Definition 2. Let y_c be a variable and k, len, h be three integers, $1 \leq len \leq |X|, 0 \leq h < len - 1$. An instantiation of $X \cup \{y_c\}$ satisfies $\text{SPRINGYFOCUS}(X, y_c, len, h, k)$ iff there exists a set S_X of disjoint sequences of indices $s_{i,j}$ such that four conditions are all satisfied: (1) $|S_X| \leq y_c$ (2) $\forall x_l \in X, x_l > k \Rightarrow \exists s_{i,j} \in S_X$ such that $l \in s_{i,j}$ (3) $\forall s_{i,j} \in S_X, j - i + 1 \leq len, x_i > k$ and $x_j > k$. (4) $\forall s_{i,j} \in S_X, |\{l \in s_{i,j}, x_l \leq k\}| \leq h$

Bounds consistency (BC) on SPRINGYFOCUS is equivalent to domain consistency: any solution can be turned into a solution that only uses the lower bound $\min(x_l)$ or the upper bound $\max(x_l)$ of the domain $D(x_l)$ of each $x_l \in X$ (this observation was made for FOCUS [Petit, 2012]). Thus, we propose a BC algorithm. The first step is to traverse X from x_0 to x_{n-1} , to compute the minimum possible number of disjoint sequences in S_X (a lower bound for y_c), the *focus cardinality*, denoted $fc(X)$. We use the same notation for subsequences of X . $fc(X)$ depends on k, len and h .

Definition 3. Given $x_l \in X$, we consider three quantities. (1) $\underline{p}(x_l, v_{\leq})$ is the focus cardinality of $[x_0, x_1, \dots, x_l]$, assuming $x_l \leq k$, and $\forall s_{i,j} \in S_{[x_0, x_1, \dots, x_l]}, j \neq l$. (2) $\underline{p}_S(x_l, v_{\leq})$ is the focus cardinality of $[x_0, x_1, \dots, x_l]$, assuming $x_l \leq k$ and $\exists i, s_{i,l} \in S_{[x_0, x_1, \dots, x_l]}$. (3) $\underline{p}(x_l, v_{>})$ is the focus cardinality of $[x_0, x_1, \dots, x_l]$ assuming $x_l > k$.

Any quantity is equal to $n + 1$ if the domain $D(x_l)$ of x_l makes not possible the considered assumption.

Property 1. $\underline{p}_S(x_0, v_{\leq}) = \underline{p}_S(x_{n-1}, v_{\leq}) = n + 1$, and $fc(X) = \min(\underline{p}(x_{n-1}, v_{\leq}), \underline{p}(x_{n-1}, v_{>}))$.

Proof. By construction from Definitions 2 and 3. \square

To compute the quantities of Definition 3 for $x_l \in X$ we use $\underline{plen}(x_l)$, the minimum length of a sequence in $S_{[x_0, x_1, \dots, x_l]}$ containing x_l among instantiations of $[x_0, x_1, \dots, x_l]$ where the number of sequences is $fc([x_0, x_1, \dots, x_l])$. $\underline{plen}(x_l) = 0$ if $\forall s_{i,j} \in S_{[x_0, x_1, \dots, x_l]}, j \neq l$. $\underline{card}(x_l)$ is the minimum number of values $v \leq k$ in the current sequence in $S_{[x_0, x_1, \dots, x_l]}$, equal to 0 if $\forall s_{i,j} \in S_{[x_0, x_1, \dots, x_l]}, j \neq l$. $\underline{card}(x_l)$ assumes that $x_l > k$. It has to be decreased by one if $x_l \leq k$. For sake of space, proofs of next lemmas are given in Appendix.

Lemma 1 (initialization). $\underline{p}(x_0, v_{\leq}) = 0$ if $\min(x_0) \leq k$, and $n + 1$ otherwise; $\underline{p}_S(x_0, v_{\leq}) = n + 1$; $\underline{p}(x_0, v_{>}) = 1$ if $\max(x_0) > k$ and $n + 1$ otherwise; $\underline{plen}(x_0) = 1$ if $\max(x_0) > k$ and 0 otherwise; $\underline{card}(x_0) = 0$.

Lemma 2 ($\underline{p}(x_l, v_{\leq})$). If $\min(x_l) \leq k$ then $\underline{p}(x_l, v_{\leq}) = \min(\underline{p}(x_{l-1}, v_{\leq}), \underline{p}(x_{l-1}, v_{>}))$, else $\underline{p}(x_l, v_{\leq}) = n + 1$.

Lemma 3 ($\underline{p}_S(x_l, v_{\leq})$). If $\min(x_l) > k$, $\underline{p}_S(x_l, v_{\leq}) = n + 1$. Otherwise, if $\underline{plen}(x_{l-1}) \in \{0, len - 1, len\} \vee \underline{card}(x_{l-1}) = h$ then $\underline{p}_S(x_l, v_{\leq}) = n + 1$, else $\underline{p}_S(x_l, v_{\leq}) = \min(\underline{p}_S(x_{l-1}, v_{\leq}), \underline{p}(x_{l-1}, v_{>}))$.

Lemma 4 ($\underline{p}(x_l, v_{>})$). If $\max(x_l) \leq k$ then $\underline{p}(x_l, v_{>}) = n + 1$. Otherwise, if $\underline{plen}(x_{l-1}) \in \{0, len\}$, $\underline{p}(x_l, v_{>}) = \min(\underline{p}(x_{l-1}, v_{>} + 1, \underline{p}(x_{l-1}, v_{\leq}) + 1)$, else $\underline{p}(x_l, v_{>}) = \min(\underline{p}(x_{l-1}, v_{>}), \underline{p}_S(x_{l-1}, v_{\leq}), \underline{p}(x_{l-1}, v_{\leq}) + 1)$.

Proposition 1 ($\underline{plen}(x_l)$). (by construction) If $\min(\underline{p}_S(x_{l-1}, v_{\leq}), \underline{p}(x_{l-1}, v_{>})) < \underline{p}(x_{l-1}, v_{\leq}) + 1 \wedge \underline{plen}(x_{l-1}) < len$ then $\underline{plen}(x_l) = \underline{plen}(x_{l-1}) + 1$. Otherwise, if $\underline{p}(x_l, v_{>}) < n + 1$ then $\underline{plen}(x_l) = 1$, else $\underline{plen}(x_l) = 0$.

Proposition 2 ($\underline{card}(x_l)$). (by construction) If $\underline{plen}(x_l) = 1$ then $\underline{card}(x_l) = 0$. Otherwise, if $\underline{p}(x_l, v_{>}) = n + 1$ then $\underline{card}(x_l) = \underline{card}(x_{l-1}) + 1$, else $\underline{card}(x_l) = \underline{card}(x_{l-1})$.

Algorithm 1: $\text{MINCARDS}(X, len, k, h)$: Integer matrix

```

1 pre := new Integer[|X|][4];
2 for l ∈ 0..n - 1 do
3   pre[l][0] := new Integer[2];
4   for j ∈ 1..3 do pre[l][j] := new Integer[1];
5 Initialization Lemma 1;
6 for l ∈ 1..n - 1 do Lemmas 2, 3, 4 and Propositions 1 and 2;
7 return pre;

```

Algorithm 1 implements the lemmas with $pre[l][0][0] = \underline{p}(x_l, v_{\leq})$, $pre[l][0][1] = \underline{p}_S(x_l, v_{\leq})$, $pre[l][1] = \underline{p}(x_l, v_{>})$, $pre[l][2] = \underline{plen}(x_l)$, $pre[l][3] = \underline{card}(x_l)$.

The principle of Algorithm 2 is the following. First, $lb = fc(X)$ is computed with x_{n-1} . We execute Algorithm 1 from x_0 to x_{n-1} and conversely (arrays *pre* and *suf*). We thus have for each quantity two values for each variable x_l . To aggregate them, we implement regret mechanisms directly derived from Propositions 2 and 1, according to the parameters len and h . Line 4 is optional but it avoids some work when the variable y_c is fixed, thanks to the same property as FOCUS (see [Petit, 2012]). Algorithm 2 performs a constant number of traversals of the set X . Its time complexity is $O(n)$, which is optimal.

3 Weighted FOCUS

We present WEIGHTEDFOCUS , that extends FOCUS with a variable z_c limiting the the sum of lengths of all the sequences in S_X , i.e., the number of variables *covered* by a sequence in S_X . It distinguishes between solutions that are equivalent with respect to the number of sequences in S_X but not with respect to their length, as Figure 2 shows.

Algorithm 2: FILTERING(X, y_c, len, k, h): Set of variables

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1  pre := MINCARDS( $X, len, k, h$ );
2  Integer lb := min(pre[n-1][0][0], pre[n-1][1]);
3  if min( $y_c$ ) < lb then  $D(y_c) := D(y_c) \setminus \{\min(y_c), lb\}$ ;
4  if min( $y_c$ ) = max( $y_c$ ) then
5    suf := MINCARDS( $[x_{n-1}, x_{n-2}, \dots, x_0], len, k, h$ );
6    for  $l \in 0..n-1$  do
7      if pre[l][0][0] + suf[n-1-l][0][0] > max( $y_c$ ) then
8        Integer regret := 0; Integer add := 0;
9        if pre[l][1] ≤ pre[l][0][1] then add := add + 1;
10       if suf[n-1-l][1] ≤ suf[n-1-l][0][1] then
11         add := add + 1;
12       if pre[l][2] + suf[n-1-l][2] - 1 ≤ len ∧
13         pre[l][3] + suf[n-1-l][3] + add - 1 ≤ h then regret
14         := 1;
15       if
16         pre[l][0][1] + suf[n-1-l][0][1] - regret > max( $y_c$ )
17         then  $D(x_l) := D(x_l) \setminus \{\min(x_l), k\}$ ;
18       Integer regret := 0;
19       if pre[l][2] + suf[n-1-l][2] - 1 ≤ len ∧
20         pre[l][3] + suf[n-1-l][3] - 1 ≤ h then regret := 1;
21       if pre[l][1] + suf[n-1-l][1] - regret > max( $y_c$ ) then
22          $D(x_l) := D(x_l) \setminus \{k, \max(x_l)\}$ ;
23     return  $X \cup \{y_c\}$ ;

```

Definition 4. Let y_c and z_c be two integer variables and k , len be two integers, such that $1 \leq len \leq |X|$. An instantiation of $X \cup \{y_c\} \cup \{z_c\}$ satisfies WEIGHTEDFOCUS(X, y_c, len, k, z_c) iff there exists a set S_X of disjoint sequences of indices $s_{i,j}$ such that four conditions are all satisfied: (1) $|S_X| \leq y_c$ (2) $\forall x_l \in X, x_l > k \Leftrightarrow \exists s_{i,j} \in S_X$ such that $l \in s_{i,j}$ (3) $\forall s_{i,j} \in S_X, j - i + 1 \leq len$ (4) $\sum_{s_{i,j} \in S_X} |s_{i,j}| \leq z_c$.

Definition 5 ([Petit, 2012]). Given an integer k , a variable $x_l \in X$ is: Penalizing, (P_k), iff $\min(x_l) > k$. Neutral, (N_k), iff $\max(x_l) \leq k$. Undetermined, (U_k), otherwise. We say $x_l \in P_k$ iff x_l is labeled P_k , and similarly for U_k and N_k .

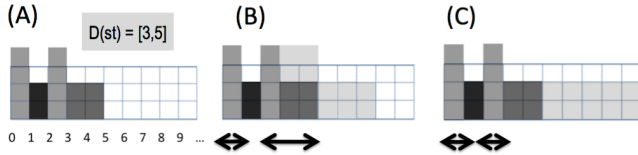


Figure 2: (A) Problem with 4 fixed activities and one activity of length 5 that can start from time 3 to 5. We assume $D(y_c) = \{2\}$, $len = 3$ and $k = 0$. (B) Solution satisfying WEIGHTEDFOCUS with $z_c = 4$. (C) Solution satisfying WEIGHTEDFOCUS with $z_c = 2$.

Dynamic Programming (DP) Principle Given a partial instantiation I_X of X and a set of sequences S_X that covers all penalizing variables in I_X , we consider two terms: the number of variables in P_k and the number of undetermined variables, in U_k , covered by S_X . We want to find a set S_X that minimizes the second term. Given a sequence of variables $s_{i,j}$, the cost $cst(s_{i,j})$ is defined as $cst(s_{i,j}) = \{p|x_p \in U_k, x_p \in s_{i,j}\}$. We denote cost of S_X , $cst(S_X)$, the sum $cst(S_X) = \sum_{s_{i,j} \in S_X} cst(s_{i,j})$. Given I_X we consider $|P_k| = |\{x_i \in P_k\}|$. We have: $\sum_{s_{i,j} \in S} |s_{i,j}| = \sum_{s_{i,j} \in S} cst(s_{i,j}) + |P_k|$.

We start with explaining the main difficulty in building a propagator for WEIGHTEDFOCUS. The constraint has two optimization variables in its scope and we might not have a solution that optimizes both variables simultaneously.

Example 1. Consider the set $X = [x_0, x_1, \dots, x_5]$ with domains $[1, \{0, 1\}, 1, 1, \{0, 1\}, 1]$ and WEIGHTEDFOCUS($X, [2, 3], 3, 0, [0, 6]$), solution $S_X = \{s_{0,2}, s_{3,5}\}$, $z_c = 6$, minimizes $y_c = 2$, while solution $S_X = \{s_{0,1}, s_{2,3}, s_{5,5}\}$, $y_c = 3$, minimizes $z_c = 4$.

Example 1 suggests that we need to fix one of the two optimization variables and only optimize the other one. Our algorithm is based on a dynamic program [Dasgupta *et al.*, 2006]. For each prefix of variables $[x_0, x_1, \dots, x_j]$ and given a cost value c , it computes a cover of focus cardinality, denoted $S_{c,j}$, which covers all penalized variables in $[x_0, x_1, \dots, x_j]$ and has cost exactly c . If $S_{c,j}$ does not exist we assume that $S_{c,j} = \infty$. $S_{c,j}$ is not unique as Example 2 demonstrates.

Example 2. Consider $X = [x_0, x_1, \dots, x_7]$ and WEIGHTEDFOCUS($X, [2, 2], 5, 0, [7, 7]$), with $D(x_i) = \{1\}$, $i \in I, I = \{0, 2, 3, 5, 7\}$ and $D(x_i) = \{0, 1\}$, $i \in \{0, 1, \dots, 7\} \setminus I$. Consider the subsequence of variables $[x_0, \dots, x_5]$ and $S_{1,5}$. There are several sets of minimum cardinality that cover all penalized variables in the prefix $[x_0, \dots, x_5]$ and has cost 2, e.g. $S_{1,5}^1 = \{s_{0,2}, s_{3,5}\}$ or $S_{1,5}^2 = \{s_{0,4}, s_{5,5}\}$. Assume we sort sequences by their starting points in each set. We note that the second set is better if we want to extend the last sequence in this set as the length of the last sequence $s_{5,5}$ is shorter compared to the length of the last sequence in $S_{1,5}^1$, which is $s_{3,5}$.

Example 2 suggests that we need to put additional conditions on $S_{c,j}$ to take into account that some sets are better than others. We can safely assume that none of the sequences in $S_{c,j}$ starts at undetermined variables as we can always set it to zero. Hence, we introduce a notion of an ordering between sets $S_{c,j}$ and define conditions that this set has to satisfy.

Ordering of sequences in $S_{c,j}$. We introduce an order over sequences in $S_{c,j}$. Given a set of sequences in $S_{c,j}$ we sort them by their starting points. We denote $last(S_{c,j})$ the last sequence in $S_{c,j}$ in this order. If $x_j \in last(S_{c,j})$ then $|last(S_{c,j})|$ is, naturally, the length of $last(S_{c,j})$, otherwise $|last(S_{c,j})| = \infty$.

Ordering of sets $S_{c,j}$, $c \in [0, \max(z_c)]$, $j \in \{0, 1, \dots, n-1\}$. We define a comparison operation between two sets $S_{c,j}$ and $S_{c',j'}$. $S_{c,j} \leq S_{c',j'}$ iff $|S_{c,j}| < |S_{c',j'}|$ or $|S_{c,j}| = |S_{c',j'}|$ and $last(S_{c,j}) \leq last(S_{c',j'})$. Note that we do not take account of cost in the comparison as the current definition is sufficient for us. Using this operation, we can compare all sets $S_{c,j}$ and $S_{c',j'}$ of the same cost for a prefix $[x_0, \dots, x_j]$. We say that $S_{c,j}$ is optimal iff satisfies the following 4 conditions.

Proposition 3 (Conditions on $S_{c,j}$).

1. $S_{c,j}$ covers all P_k variables in $[x_0, x_1, \dots, x_j]$,
2. $cst(S_{c,j}) = c$,
3. $\forall s_{h,g} \in S_{c,j}, x_h \notin U_k$,
4. $S_{c,j}$ is the first set in the order among all sets that satisfy conditions 1–3.

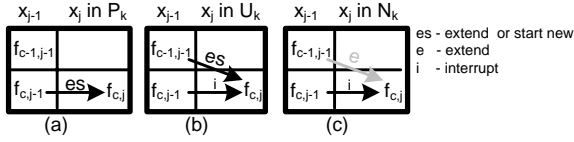


Figure 3: Representation of one step of Algorithm 3.

As can be seen from definitions above, given a subsequence of variables x_0, \dots, x_j , $S_{c,j}$ is not unique and might not exist. However, if $|S_{c,j}| = |S_{c',j'}|$, $c = c'$ and $j = j'$, then $last(S_{c,j}) = last(S_{c',j'})$.

Example 3. Consider WEIGHTEDFOCUS from Example 2. Consider the subsequence $[x_0, x_1]$. $S_{0,1} = \{s_{0,0}\}$, $S_{1,1} = \{s_{0,1}\}$. Note that $S_{2,1}$ does not exist. Consider the subsequence $[x_0, \dots, x_5]$. We have $S_{0,5} = \{s_{0,0}, s_{2,3}, s_{5,5}\}$, $S_{1,5} = \{s_{0,4}, s_{5,5}\}$ and $S_{2,5} = \{s_{0,3}, s_{5,5}\}$. By definition, $last(S_{0,5}) = s_{5,5}$, $last(S_{1,5}) = s_{5,5}$ and $last(S_{2,5}) = s_{5,5}$. Consider the set $S_{1,5}$. Note that there exists another set $S'_{1,5} = \{s_{0,0}, s_{2,5}\}$ that satisfies conditions 1–3. Hence, it has the same cardinality as $S_{1,5}$ and the same cost. However, $S_{1,5} < S'_{1,5}$ as $|last(S_{1,5})| = 1 < |last(S'_{1,5})| = 3$.

Bounds disentanglement Each cell in the dynamic programming table $f_{c,j}$, $c \in [0, z_c^U]$, $j \in \{0, 1, \dots, n-1\}$, where $z_c^U = max(z_c) - |P_k|$, is a pair of values $q_{c,j}$ and $l_{c,j}$, $f_{c,j} = \{q_{c,j}, l_{c,j}\}$, stores information about $S_{c,j}$. Namely, $q_{c,j} = |S_{c,j}|$, $l_{c,j} = |last(S_{c,j})|$ if $last(S_{c,j}) \neq \infty$ and ∞ otherwise. We say that $f_{c,j}/q_{c,j}/l_{c,j}$ is a dummy (takes a dummy value) iff $f_{c,j} = \{\infty, \infty\}/q_{c,j} = \infty/l_{c,j} = \infty$. If $y_1 = \infty$ and $y_2 = \infty$ then we assume that they are equal. We introduce a dummy variable x_{-1} , $D(x_{-1}) = \{0\}$ and a row $f_{-1,j}$, $j = -1, \dots, n-1$ to keep uniform notations.

Algorithm 3: Weighted FOCUS(x_0, \dots, x_{n-1})

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1 for  $c \in -1..z_c^U$  do
2   for  $j \in -1..n-1$  do
3      $f_{c,j} \leftarrow \{\infty, \infty\}$ ;
4    $f_{-1,-1} \leftarrow \{0, 0\}$ ;
5   for  $j \in 0..n-1$  do
6     for  $c \in 0..j$  do
7       if  $x_j \in P_k$  then /* penalizing */
8         if  $(l_{c,j-1} \in [1, len]) \vee (q_{c,j-1} = \infty)$  then
9            $f_{c,j} \leftarrow \{q_{c,j-1}, l_{c,j-1} + 1\}$ ;
10          else  $f_{c,j} \leftarrow \{q_{c,j-1} + 1, 1\}$ ;
11        if  $x_j \in U_k$  then /* undetermined */
12          if  $(l_{c-1,j-1} \in [1, len] \wedge q_{c-1,j-1} =$ 
13              $q_{c,j-1}) \vee (q_{c,j-1} = \infty)$  then
14             $f_{c,j} \leftarrow \{q_{c-1,j-1}, l_{c-1,j-1} + 1\}$  else
15             $f_{c,j} \leftarrow \{q_{c,j-1}, \infty\}$ 
16          if  $x_j \in N_k$  then /* neutral */
17             $f_{c,j} \leftarrow \{q_{c,j-1}, \infty\}$ 
18  return  $f$ ;
```

Algorithm 3 gives pseudocode for the propagator. The intuition behind the algorithm is as follows. We emphasize again that by cost we mean the number of covered variables in U_k .

If $x_j \in P_k$ then we do not increase the cost of $S_{c,j}$ compared to $S_{c,j-1}$ as the cost only depends on $x_j \in U_k$. Hence, the best move for us is to extend $last(S_{c,j-1})$ or start a new sequence if it is possible. This is encoded in lines 9 and 10 of

c	$D(x_0)$	$D(x_1)$	$D(x_2)$	$D(x_3)$	$D(x_4)$	$D(x_5)$	$D(x_6)$	$D(x_7)$
0	{1, 1}	{1, ∞}	{2, 1}	{2, 2}	{2, ∞}	{3, 1}	{3, ∞}	{4, 1}
1	{1, 1}	{1, 2}	{1, 3}	{1, 4}	{1, ∞}	{2, 1}	{2, ∞}	{3, 1}
$z_c^U = 2$					{1, 5}	{2, 1}	{2, 2}	{2, 3}

Table 1: An execution of Algorithm 3 on WEIGHTEDFOCUS from Example 2. Dummy values $f_{c,j}$ are removed.

the algorithm. Figure 3(a) gives a schematic representation of these arguments.

If $x_j \in U_k$ then we have two options. We can obtain $S_{c,j}$ from $S_{c-1,j-1}$ by increasing $cost(S_{c-1,j-1})$ by one. This means that x_i will be covered by $last(S_{c,j})$. Alternatively, from $S_{c,j-1}$ by interrupting $last(S_{c,j-1})$. This is encoded in line 12 of the algorithm (Figure 3(b)).

If $x_j \in N_k$ then we do not increase the cost of $S_{c,j}$ compared to $S_{c,j-1}$. Moreover, we must interrupt $last(S_{c,j-1})$, line 14 (Figure 3(c), ignore the gray arc).

First we prove a property of the dynamic programming table. We define a comparison operation between $f_{c,j}$ and $f_{c',j'}$ induced by a comparison operation between $S_{c,j}$ and $S_{c',j'}$: $f_{c,j} \leq f_{c',j'}$ if $(q_{c,j} < q_{c',j'})$ or $(q_{c,j} = q_{c',j'})$ and $l_{c,j} \leq l_{c',j'}$. In other words, as in a comparison operation between sets, we compare by the cardinality of sequences, $|S_{c,j}|$ and $|S_{c',j'}|$, and, then by the length of the last sequence in each set, $last(S_{c,j})$ and $last(S_{c',j'})$. We omit proofs of the next two lemmas due to space limitations (see Appendix).

Lemma 5. Consider WEIGHTEDFOCUS(X, y_c, len, k, z_c). Let f be dynamic programming table returned by Algorithm 3. Non-dummy elements $f_{c,j}$ are monotonically non-increasing in each column, so that $f_{c',j} \leq f_{c,j}$, $0 \leq c < c' \leq z_c^U$, $j \in [0, \dots, n-1]$.

Lemma 6. Consider WEIGHTEDFOCUS(X, y_c, len, k, z_c). The dynamic programming table $f_{c,j} = \{q_{c,j}, l_{c,j}\}$ $c \in [0, z_c^U]$, $j = 0, \dots, n-1$, is correct in the sense that if $f_{c,j}$ exists and it is non-dummy then a corresponding set of sequences $S_{c,j}$ exists and satisfies conditions 1–4. The time complexity of Algorithm 3 is $O(n \max(z_c))$.

Example 4. Table 1 shows an execution of Algorithm 3 on WEIGHTEDFOCUS from Example 2. Note that $|P_0| = 5$. Hence, $z_c^U = max(z_c) - |P_0| = 2$. As can be seen from the table, the constraint has a solution as there exists a set $S_{2,7} = \{s_{0,3}, s_{5,7}\}$ such that $|S_{2,7}| = 2$.

Bounds consistency To enforce BC on variables x , we compute an additional DP table b , $b_{c,j}$, $c \in [0, z_c^U]$, $j \in [-1, n-1]$ on the reverse sequence of variables x .

Lemma 7. Consider WEIGHTEDFOCUS(X, y_c, len, k, z_c). Bounds consistency can be enforced in $O(n \max(z_c))$ time.

Proof. (Sketch) We build dynamic programming tables f and b . We will show that to check if $x_i = v$ has a support it is sufficient to examine $O(z_c^U)$ pairs of values $f_{c_1, i-1}$ and $b_{c_2, n-i-2}$, $c_1, c_2 \in [0, z_c^U]$ which are neighbor columns to the i th column. It is easy to show that if we consider all possible pairs of elements in $f_{c_1, i-1}$ and $b_{c_2, n-i-2}$ then we determine if there exists a support for $x_i = v$. There are $O(z_c^U \times z_c^U)$

such pairs. The main part of the proof shows that it sufficient to consider $O(z_c^U)$ such pairs. In particular, to check a support for a variable-value pair $x_i = v$, $v > k$, for each $f_{c_1, i-1}$ it is sufficient to consider only one element $b_{c_2, n-i-2}$ such that $b_{c_2, n-i-2}$ is non-dummy and c_2 is the maximum value that satisfies inequality $c_1 + c_2 + 1 \leq z_c^U$. To check a support for a variable-value pair $x_i = v$, $v \leq k$, for each $f_{c_1, i-1}$ it is sufficient to consider only one element $b_{c_2, n-i-2}$ such that $b_{c_2, n-i-2}$ is non-dummy and c_2 is the maximum value that satisfies inequality $c_1 + c_2 \leq z_c^U$. \square

We observe a useful property of the constraint. If there exists $f_{c, n-1}$ such that $c < \max(z_c)$ and $q_{c, n-1} < \max(y_c)$ then the constraint is BC. This follows from the observation that given a solution of the constraint S_X , changing a variable value can increase $cst(S_X)$ and $|S_X|$ by at most one.

Alternatively we can decompose WEIGHTEDFOCUS using $O(n)$ additional variables and constraints.

Proposition 4. *Given FOCUS(X, y_c, len, k), let z_c be a variable and $B = [b_0, b_1, \dots, b_{n-1}]$ be a set of variables such that $\forall b_l \in B, D(b_l) = \{0, 1\}$. $\text{WEIGHTEDFOCUS}(X, y_c, len, k, z_c) \Leftrightarrow \text{FOCUS}(X, y_c, len, k) \wedge [\forall l, 0 \leq l < n, [(x_l \leq k) \wedge (b_l = 0)] \vee [(x_l > k) \wedge (b_l = 1)]] \wedge \sum_{l \in \{0, 1, \dots, n-1\}} b_l \leq z_c$.*

Enforcing BC on each constraint of the decomposition is weaker than BC on WEIGHTEDFOCUS. Given $x_l \in X$, a value may have a unique support for FOCUS which violates $\sum_{l \in \{0, 1, \dots, n-1\}} b_l \leq z_c$, and conversely. Consider $n=5$, $x_0=x_2=1$, $x_3=0$, and $D(x_1)=D(x_4)=\{0, 1\}$, $y_c=2$, $z_c=3$, $k=0$ and $len=3$. Value 1 for x_4 corresponds to this case.

4 Weighted Springy FOCUS

We consider a further generalization of the FOCUS constraint that combines SPRINGYFOCUS and WEIGHTEDFOCUS. We prove that we can propagate this constraint in $O(n \max(z_c))$ time, which is same as enforcing BC on WEIGHTEDFOCUS.

Definition 6. *Let y_c and z_c be two variables and k, len, h be three integers, such that $1 \leq len \leq |X|$ and $0 < h < len - 1$. An instantiation of $X \cup \{y_c\} \cup z_c$ satisfies WEIGHTEDSPRINGYFOCUS(X, y_c, len, h, k, z_c) iff there exists a set S_X of disjoint sequences of indices $s_{i,j}$ such that five conditions are all satisfied: (1) $|S_X| \leq y_c$ (2) $\forall x_l \in X, x_l > k \Rightarrow \exists s_{i,j} \in S_X$ such that $l \in s_{i,j}$ (3) $\forall s_{i,j} \in S_X, |\{l \in s_{i,j}, x_l \leq k\}| \leq h$ (4) $\forall s_{i,j} \in S_X, j - i + 1 \leq len, x_i > k$ and $x_j > k$. (5) $\sum_{s_{i,j} \in S_X} |s_{i,j}| \leq z_c$.*

We can again partition cost of S into two terms. $\sum_{s_{i,j} \in S} |s_{i,j}| = \sum_{s_{i,j} \in S} cst(s_{i,j}) + |P_k|$. However, $cst(s_{i,j})$ is the number of *undetermined* and *neutral* variables covered $s_{i,j}$, $cst(s_{i,j}) = \{p | x_p \in U_k \cup N_k, x_p \in s_{i,j}\}$ as we allow to cover up to h neutral variables.

The propagator is again based on a dynamic program that for each prefix of variables $[x_0, x_1, \dots, x_j]$ and given cost c computes a cover $S_{c,j}$ of minimum cardinality that covers all penalized variables in the prefix $[x_0, x_1, \dots, x_j]$ and has cost *exactly* c . We face the same problem of how to compare two sets $S_{c_1, j}^1$ and $S_{c_2, j}^2$ of minimum cardinality. The issue here is how to compare $last(S_{c_1, j}^1)$ and $last(S_{c_2, j}^2)$ if they

cover a different number of neutral variables. Luckily, we can avoid this problem due to the following monotonicity property. If $last(S_{c_1, j}^1)$ and $last(S_{c_2, j}^2)$ are not equal to infinity then they both end at the same position j . Hence, if $last(S_{c_1, j}^1) \leq last(S_{c_2, j}^2)$ then the number of neutral variables covered by $last(S_{c_1, j}^1)$ is no larger than the number of neutral variables covered by $last(S_{c_2, j}^2)$. Therefore, we can define order on sets $S_{c,j}$ as we did in Section 3 for WEIGHTEDFOCUS.

Our bounds disentailment detection algorithm for WEIGHTEDSPRINGYFOCUS mimics Algorithm 3. We omit the pseudocode due to space limitations but highlight two not-trivial differences between this algorithm and Algorithm 3. The first difference is that each cell in the dynamic programming table $f_{c,j}$, $c \in [0, z_c^U]$, $j \in \{0, 1, \dots, n-1\}$, where $z_c^U = \max(z_c) - |P_k|$, is a triple of values $q_{c,j}$, $l_{c,j}$ and $h_{c,j}$, $f_{c,j} = \{q_{c,j}, l_{c,j}, h_{c,j}\}$. The new parameter $h_{c,j}$ stores the number of neutral variables covered by $last(S_{c,j})$. The second difference is in the way we deal with neutral variables. If $x_j \in N_k$ then we have two options now. We can obtain $S_{c,j}$ from $S_{c-1, j-1}$ by increasing $cst(S_{c-1, j-1})$ by one and increasing the number of covered neutral variables by $last(S_{c, j-1})$ (Figure 3(c), the gray arc). Alternatively, we can obtain $S_{c,j}$ from $S_{c, j-1}$ by interrupting $last(S_{c, j-1})$ (Figure 3(c), the black arc). BC can enforced using two modifications of the corresponding algorithm for WEIGHTEDFOCUS (a proof is given in Appendix).

Lemma 8. *Consider WEIGHTEDSPRINGYFOCUS(X, y_c, len, h, k, z_c). BC can be enforced in $O(n \max(z_c))$ time.*

WEIGHTEDSPRINGYFOCUS can be encoded using the cost-REGULAR constraint. The automaton need 3 counters to compute len, y_c and h . Hence, the time complexity of this encoding is $O(n^4)$. This automaton is non-deterministic as on seeing $v \leq k$, it either covers the variable or interrupts the last sequence. Unfortunately the non-deterministic cost-REGULAR is not implemented in any constraint solver to our knowledge. In contrast, our algorithm takes just $O(n^2)$ time. WEIGHTEDSPRINGYFOCUS can also be decomposed using the GCC constraint [Régis, 1996]. We define the following variables for all $i \in [0, \max(y_c) - 1]$ and $j \in [0, n-1]$: S_i the start of the i th sub-sequence. $D(S_i) = \{0, \dots, n + \max(y_c)\}$; E_i the end of the i th sub-sequence. $D(E_i) = \{0, \dots, n + \max(y_c)\}$; T_j the index of the subsequence in S_X containing x_j . $D(T_j) = \{0, \dots, \max(y_c)\}$; Z_j the index of the subsequence in S_X containing x_j s.t. the value of x_j is less than or equal to k . $D(Z_j) = \{0, \dots, \max(y_c)\}$; $last_c$ the cardinality of S_X . $D(last_c) = \{0, \dots, \max(y_c)\}$; $Card$, a vector of $\max(y_c)$ variables having $\{0, \dots, h\}$ as domains. $\text{WEIGHTEDSPRINGYFOCUS}(X, y_c, len, h, k, z_c) \Leftrightarrow$

$$\begin{aligned}
& (x_j \leq k) \vee Z_j = 0; & (x_j \leq k) \vee T_j > 0; \\
& (x_j > k) \vee (T_j = Z_j); & (T_j \leq last_c); \\
& (T_j \neq i) \vee (j \geq S_{i-1}); & (T_j \neq i) \vee (j \leq E_{i-1}); \\
& (i > last_c) \vee (T_j = i) \vee (j < S_{i-1}) \vee (j > E_{i-1}); \\
& \forall q \in [1, \max(y_c) - 1], & q \geq last_c \vee S_q > E_{q-1}; \\
& \forall q \in [0, \max(y_c) - 1] & q \geq last_c \vee E_q \geq S_q; \\
& \forall q \in [0, \max(y_c) - 1] & q \geq last_c \vee len > (E_q - S_q); \\
& last_c \leq y_c; & Gcc([T_0, \dots, T_{n-1}], \{0\}, [n - z_c]); \\
& & Gcc([Z_0, \dots, Z_{n-1}], \{1, \dots, \max(y_c)\}, Card);
\end{aligned}$$

Table 2: SLS with WEIGHTEDFOCUS and its decomposition.

		16.1			16.2			16.3		
	#n	#b	T	#n	#b	T	#n	#b	T	
F	50	0.9K	0	50	4.1K	2	47	18.1K	7	
D ₁	50	3.4K	1	49	8.1K	3	44	21.8K	8	

		20.1			20.2			20.3		
	#n	#b	T	#n	#b	T	#n	#b	T	
F	49	11.8K	7	45	24.9K	14	39	36.5K	23	
D ₁	43	30.8K	13	35	27.2K	12	29	29.6K	17	

5 Experiments

We used the Choco-2.1.5 solver on an IntelXeon 2.27GHz for the first benchmarks and IntelXeon 3.20GHz for last ones, both under Linux. We compared the propagators (denoted by F) of WEIGHTEDFOCUS and WEIGHTEDSPRINGYFOCUS against two decompositions (denoted by D₁ and D₂), using the same search strategies, on three different benchmarks. The first decomposition, restricted to WEIGHTEDFOCUS, is shown in proposition 4, while the second one is shown in Section 4. In the tables, we report for each set the total number of solved instances (#n), then we average both the number of backtracks (#b) and the resolution time (T) in seconds.

□ *Sports league scheduling (SLS)*. We extend a single round-robin problem with $n = 2p$ teams. Each week each team plays a game either at home or away. Each team plays exactly once all the other teams during a season. We minimize the number of breaks (a break for one team is two consecutive home or two consecutive away games), while fixed weights in $\{0, 1\}$ are assigned to all games: games with weight 1 are important for TV channels. The goal is to group consecutive weeks where at least one game is important (sum of weights > 0), to increase the price of TV broadcast packages. Packages are limited to 5 weeks and should be as short as possible. Table 2 shows results with 16 and 20 teams, on sets of 50 instances with 10 random important games and a limit of 400K backtracks. $\max(y_c) = 3$ and we search for one solution with $h \leq 7$ (instances $n-1$), $h \leq 6$ ($n-2$) and $h \leq 5$ ($n-3$). In our model, inverse-channeling and ALLDIFFERENT constraints with the strongest propagation level express that each team plays once against each other team. We assign first the sum of breaks by team, then the breaks and places using the *DomOverWDeg* strategy.

□ *Cumulative Scheduling with Rentals*. Given a horizon of n days and a set of time intervals $[s_i, e_i]$, $i \in \{1, 2, \dots, p\}$, a company needs to rent a machine between l_i and u_i times within each time interval $[s_i, e_i]$. We assume that the cost of the rental period is proportional to its length. On top of this, each time the machine is rented we pay a fixed cost. The problem is then defined as a conjunction of one WEIGHTEDSPRINGYFOCUS($X, y_c, len, h, 0, z_c$) with a set of AMONG constraints. The goal is to build a schedule for rentals that satisfies all demand constraints and minimizes simultaneously the number of rental periods and their total length. We build a Pareto frontier over two cost variables, as Figure 4 shows for one of the instances of this problem. We generated instances having a fixed length of sub-sequences of size 20 (i.e., $len = 20$), 50% as a probability of posting an *Among* constraint for each (i, j) s.t. $j \geq i+5$ in the sequence.

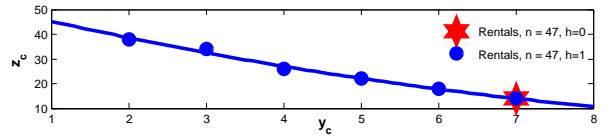


Figure 4: Pareto frontier for Scheduling with Rentals.

Table 3: Scheduling with Rentals.

		40			43			45		
h		#n	#b	T	#n	#b	T	#n	#b	T
0	F	20	349K	55.4	20	1M	192.2	20	1M	233.7
0	D ₁	20	529K	74.7	20	1M	251.2	20	1M	328.6
1	F	20	827M	120.4	20	2M	420.9	19	3M	545.9
2	F	20	826K	115.7	20	2M	427.3	19	3M	571.3

		47			50		
h		#n	#b	T	#n	#b	T
0	F	19	1M	354.5	18	2M	553.7
0	D ₁	18	2M	396.8	17	3M	660
1	F	16	4M	725.4	4	6M	984.5
2	F	15	4M	763.9	4	5M	944.8

Each set of instances corresponds to a unique sequence size ($\{40, 43, 45, 47, 50\}$) and 20 different seeds. We summarize these tests in table 3. Results with decomposition are very poor. We therefore consider only the propagator in this case.

□ *Sorting Chords*. We need to sort n distinct chords. Each chord is a set of at most p notes played simultaneously. The goal is to find an ordering that minimizes the number of notes changing between two consecutive chords. The full description and a CP model is in [Petit, 2012]. The main difference here is that we build a Pareto frontier over two cost variables. We generated 4 sets of instances distinguished by the numbers of chords ($\{14, 16, 18, 20\}$). We fixed the length of the subsequences and the maximum notes for all the sets then change the seed for each instance.

Tables 2, 3 and 4 show that best results were obtained with our propagators (number of solved instances, average backtracks and CPU time over all the solved instances¹). Figure 4 confirms the gain of flexibility illustrated by Figure 1 in Section 2: allowing $h = 1$ variable with a low cost value into each sequence leads to new solutions, with significantly lower values for the target variable y_c .

6 Conclusion

We have presented flexible tools for capturing the concept of concentrating costs. Our contribution highlights the expressive power of constraint programming, in comparison with other paradigms where such a concept would be very difficult to represent. Our experiments have demonstrated the effectiveness of the proposed new filtering algorithms.

¹While the technique that solves the largest number of instances (and thus some harder ones) should be penalized.

Table 4: Sorting Chords

		14			16			18			20		
h		#n	#b	T	#n	#b	T	#n	#b	T	#n	#b	T
0	F	30	70K	2.8	30	865K	14.6	28	10M	182.9	16	14M	270.4
0	D ₁	30	94K	3.2	30	2M	41	28	12M	206.9	13	10M	206.8
0	D ₂	30	848K	34.9	24	3M	122.3	13	8M	285.6	7	902K	38.7
1	F	30	97K	3.5	30	1K	27.2	28	12K	214.2	14	13M	288.2
1	D ₂	30	851M	41.5	23	2M	116.3	11	5M	209.9	7	868K	41.5
2	F	30	97K	3.4	30	1M	25.9	28	13M	217.4	13	12M	245.5
2	D ₂	30	844K	40.9	24	3M	145.1	12	6K	251.6	7	867K	42.8

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Appendix

Proofs of Lemmas 1 to 4 omit the obvious cases where quantities take the default value $n + 1$.

A Proof of Lemma 1

From item 4 of Definition 2, a sequence in S_X cannot start with a value $v \leq k$. Thus, $\underline{p}_S(x_0, v_{\leq}) = n + 1$ and $\underline{card}(x_0) = 0$. If x_0 can take a value $v > k$ then by Definition 3, $\underline{p}(x_0, v_{>}) = 1$ and $\underline{plen}(x_0) = 1$. \square

B Proof of Lemma 2

If $\min(x_l) \leq k$ then $\underline{p}_S(x_{l-1}, v_{\leq})$ must not be considered: it would imply that a sequence in S_X ends by a value $v \leq k$ for x_{l-1} . From Property 1, the focus cardinality of the previous sequence is $\min(\underline{p}(x_{l-1}, v_{\leq}), \underline{p}(x_{l-1}, v_{>}))$. \square

C Proof of Lemma 3

If $\min(x_i) \leq k$ we have three cases to consider. (1) If either $\underline{plen}(x_{i-1}) = 0$ or $\underline{plen}(x_{i-1}) = len$ then from item 3 of Definition 2 a sequence in S_X cannot start with a value $v_i \leq k$: $\underline{p}_S(x_i, v_{\leq}) = n + 1$. (2) If $\underline{plen}(x_{i-1}) = len - 1$ then from Definition 2 the current variable x_i cannot end the sequence with a value $v_i \leq k$. (3) Otherwise, from item 3 of Definition 2, $\underline{p}(x_{i-1}, v_{\leq})$ is not considered. Thus, from Property 1, $\underline{p}_S(x_i, v_{\leq}) = \min(\underline{p}_S(x_{i-1}, v_{\leq}), \underline{p}(x_{i-1}, v_{>}))$. \square

D Proof of Lemma 4

If $\underline{plen}(x_{l-1}) \in \{0, len\}$ a new sequence has to be considered: $\underline{p}_S(x_{l-1}, v_{\leq})$ must not be considered, from item 3 of Definition 2. Thus, $\underline{p}(x_l, v_{>}) = \min(\underline{p}(x_{l-1}, v_{>}) + 1, \underline{p}(x_{l-1}, v_{\leq}) + 1)$. Otherwise, either a new sequence has to be considered ($\underline{p}(x_{l-1}, v_{\leq}) + 1$) or the value is equal to the focus cardinality of the previous sequence ending in x_{l-1} . \square

E Proof of Lemma 5

First, we prove two technical results.

Lemma 9. Consider $\text{WEIGHTEDFOCUS}([x_0, \dots, x_{n-1}], y_c, len, k, z_c)$. Let f be dynamic programming table returned by Algorithm 3. Then the non-dummy values of $f_{c,j}$ are consecutive in each column, so that there do not exist c, c', c'' , $0 \leq c < c' < c'' \leq z_c^U$, such that $f_{c',j}$ is dummy and $f_{c,j}, f_{c'',j}$ are non-dummy.

Proof. We prove by induction on the length of the sequence. The base case is trivial as $f_{0,-1} = \{0, 0\}$ and $f_{c,-1} = \{\infty, \infty\}$, $c \in [-1] \cup [1, z_c^U]$. Suppose the statement holds for $j - 1$ variables.

Suppose there exist c, c', c'' , $0 \leq c < c' < c'' \leq z_c^U$, such that $f_{c',j}$ is dummy and $f_{c,j}, f_{c'',j}$ are non-dummy.

Case 1. Consider the case $x_j \in P_k$. By Algorithm 3, lines 9 and 10, $q_{c,j} \in [q_{c,j-1}, q_{c,j-1} + 1]$, $q_{c',j} \in [q_{c',j-1}, q_{c',j-1} + 1]$ and $q_{c'',j} \in [q_{c'',j-1}, q_{c'',j-1} + 1]$. As $f_{c',j}$ is dummy and $f_{c,j}, f_{c'',j}$ are non-dummy, $f_{c',j-1}$ must

be dummy and $f_{c,j-1}, f_{c',j-1}$ must be non-dummy. This violates induction hypothesis.

Case 2. Consider the case $x_j \in U_k$. By Algorithm 3, lines 12 and 12, $q_{c,j} = \min(q_{c-1,j-1}, q_{c,j-1})$, $q_{c',j} = \min(q_{c'-1,j-1}, q_{c',j-1})$ and $q_{c'',j} = \min(q_{c''-1,j-1}, q_{c'',j-1})$. As $f_{c,j}$ is dummy, then both $f_{c-1,j-1}$ and $f_{c',j-1}$ must be dummy. As $f_{c,j}$ is non-dummy, then one of $f_{c-1,j-1}$ and $f_{c,j-1}$ is non-dummy. As $f_{c',j}$ is non-dummy, then one of $f_{c''-1,j-1}$ and $f_{c'',j-1}$ is non-dummy. We know that $c-1 < c \leq c'-1 < c' \leq c''-1 < c''$ or $c < c' < c''$. This leads to violation of induction hypothesis.

Case 3. Consider the case $x_j \in N_k$. By Algorithm 3, line 14, $q_{c,j} = q_{c,j-1}$, $q_{c',j} = q_{c',j-1}$ and $q_{c'',j} = q_{c'',j-1}$. Hence, $f_{c',j-1}$ is dummy and $f_{c,j-1}, f_{c'',j-1}$ are non-dummy. This leads to violation of induction hypothesis. \square

Proposition 5. Consider $\text{WEIGHTEDFOCUS}([x_0, \dots, x_{n-1}], y_c, \text{len}, k, z_c)$. Let f be dynamic programming table returned by Algorithm 3. The elements of the first row are non-dummy: $f_{0,j}$, $j = -1, \dots, n$ are non-dummy.

Proof. We prove by induction on the length of the sequence. The base case is trivial as $f_{0,-1} = \{0, 0\}$. Suppose the statement holds for $j-1$ variables.

Case 1. Consider the case $x_j \in P_k$. As $f_{0,j-1}$ is non-dummy then by Algorithm 3, lines 9–10, $f_{0,j}$ is non-dummy.

Case 2. Consider the case $x_j \in U_k$. Consider the condition $(l_{-1,j-1} \in [1, \text{len}] \wedge q_{-1,j-1} = q_{0,j-1}) \vee (q_{0,j-1} = \infty)$ at line 12. By the induction hypothesis, $q_{0,j-1} \neq \infty$. By the initialization procedure of the dummy row, $q_{-1,j-1} = \infty$. Hence, this condition does not hold and, by line 12, $f_{0,j}$ is non-dummy.

Case 3. Consider the case $x_j \in N_k$. As $f_{0,j-1}$ is non-dummy then by Algorithm 3, lines 14, $f_{0,j}$ is non-dummy. \square

We can now prove Lemma 5.

Proof. By transitivity and consecutivity of non-dummy values (Lemma 9) and the result that all elements in the 0th row are non-dummy (Proposition 5), it is sufficient to consider the case $c' = c + 1$.

We prove by induction on the length of the sequence. The base case is trivial as $f_{0,-1} = \{0, 0\}$ and $f_{c,0}$ are dummy, $c \in [0, z_c^U]$. Suppose the statement holds for $j-1$ variables.

Consider the variable x_j . Suppose, by contradiction, that $f_{c,j} < f_{c+1,j}$. Then either $q_{c,j} < q_{c+1,j}$ or $q_{c,j} = q_{c+1,j}, l_{c,j} < l_{c+1,j}$. By induction hypothesis, we know that $f_{c,j-1} \geq f_{c+1,j-1}$, hence, either $q_{c,j-1} > q_{c+1,j-1}$ or $q_{c,j-1} = q_{c+1,j-1}, l_{c,j-1} \geq l_{c+1,j-1}$.

We consider three cases depending on whether x_j is a penalizing variable, an undetermined variable or a neutral variable.

Case 1. Consider the case $x_j \in P_k$. If $q_{c,j-1} = \infty$ then $q_{c+1,j-1} = \infty$ by the induction hypothesis. Hence, by Algorithm 3, line 9, $f_{c,j}$ and $f_{c+1,j}$ are dummy and equal. Suppose $q_{c,j-1} \neq \infty$. Then we consider four cases based on relative values of $q_{c,j'}, q_{c+1,j'}, l_{c,j'}, l_{c+1,j'}, j' \in \{j-1, j\}$.

• **Case 1a.** Suppose $q_{c,j} < q_{c+1,j}$ and $q_{c,j-1} > q_{c+1,j-1}$. By Algorithm 3, lines 9 and 10, $q_{c,j} \geq q_{c,j-1}$ and $q_{c+1,j} \leq q_{c+1,j-1} + 1$. Hence, $q_{c,j} < q_{c+1,j}$ implies $q_{c+1,j-1} < q_{c,j} < q_{c+1,j-1} + 1$. We derive a contradiction.

• **Case 1b.** Suppose $q_{c,j} < q_{c+1,j}$ and $q_{c,j-1} = q_{c+1,j-1}, l_{c,j-1} \geq l_{c+1,j-1}$. By Algorithm 3, lines 9 and 10, $q_{c,j} \geq q_{c,j-1}$ and $q_{c+1,j} \leq q_{c+1,j-1} + 1$. Hence, $q_{c,j} < q_{c+1,j}$ implies $q_{c+1,j-1} = q_{c,j-1} \leq q_{c,j} < q_{c+1,j} \leq q_{c+1,j-1} + 1$. Hence, $q_{c+1,j-1} = q_{c,j-1} = q_{c,j}$ and $q_{c+1,j} = q_{c+1,j-1} + 1$. As $q_{c,j-1} = q_{c,j}$ then $l_{c,j-1} \in [1, \text{len})$ by Algorithm 3 line 9. As $q_{c+1,j} = q_{c+1,j-1} + 1$ then $l_{c+1,j-1} \in \{\text{len}, \infty\}$ by Algorithm 3 line 10. This leads to a contradiction as $l_{c,j-1} \geq l_{c+1,j-1}$.

• **Case 1c.** Suppose $q_{c,j} = q_{c+1,j}, l_{c,j} < l_{c+1,j}$ and $q_{c,j-1} > q_{c+1,j-1}$. Symmetric to Case 1b.

• **Case 1d.** Suppose $q_{c,j} = q_{c+1,j}, l_{c,j} < l_{c+1,j}$ and $q_{c,j-1} = q_{c+1,j-1}, l_{c,j-1} \geq l_{c+1,j-1}$. By Algorithm 3, lines 9 and 10, $q_{c,j} \geq q_{c,j-1}$ and $q_{c+1,j} \leq q_{c+1,j-1} + 1$. Hence, $q_{c,j} = q_{c+1,j}$ implies $q_{c+1,j-1} = q_{c,j-1} \leq q_{c,j} = q_{c+1,j} \leq q_{c+1,j-1} + 1$. Therefore, either $q_{c,j} = q_{c,j-1} \wedge q_{c+1,j} = q_{c+1,j-1}$ or $q_{c,j} = q_{c,j-1} + 1 \wedge q_{c+1,j} = q_{c+1,j-1} + 1$.

If $q_{c,j} = q_{c,j-1}$ and $q_{c+1,j} = q_{c+1,j-1}$ then $l_{c,j-1} \in [1, \text{len})$ and $l_{c+1,j-1} \in [1, \text{len})$ by Algorithm 3 line 9. Hence, $l_{c,j} = l_{c,j-1} + 1$ and $l_{c+1,j} = l_{c+1,j-1} + 1$. As $l_{c,j-1} \geq l_{c+1,j-1}$, then $l_{c,j} \geq l_{c+1,j}$. This leads to a contradiction with the assumption $l_{c,j} < l_{c+1,j}$.

If $q_{c,j} = q_{c,j-1} + 1 \wedge q_{c+1,j} = q_{c+1,j-1} + 1$ then $l_{c,j-1} \in \{\text{len}, \infty\}$ and $l_{c+1,j-1} \in [1, \text{len}, \infty)$ by Algorithm 3 line 10. Hence, $l_{c,j} = 1$ and $l_{c+1,j} = 1$. This leads to a contradiction with the assumption $l_{c,j} < l_{c+1,j}$.

Case 2. Consider the case $x_j \in U_k$. If $q_{c,j-1} = \infty$ then $q_{c+1,j-1} = \infty$ by the induction hypothesis. Hence, by Algorithm 3, line 12, $f_{c,j}$ and $f_{c+1,j}$ are dummy and equal.

Suppose $q_{c,j-1} \neq \infty$. Then we consider four cases based on relative values of $q_{c,j'}, q_{c+1,j'}, l_{c,j'}, l_{c+1,j'}, j' \in \{j-1, j\}$.

• **Case 2a** Suppose $q_{c,j} < q_{c+1,j}$ and $q_{c,j-1} > q_{c+1,j-1}$. By Algorithm 3, lines 12 and 12, we know that $q_{c+1,j-1} \leq q_{c+1,j} \leq q_{c,j-1}$ and $q_{c,j-1} \leq q_{c,j} \leq q_{c-1,j-1}$. By induction hypothesis, $q_{c+1,j-1} \leq q_{c,j-1} \leq q_{c-1,j-1}$. Hence, if $q_{c,j} \leq q_{c+1,j}$ then $q_{c,j-1} \leq q_{c,j} \leq q_{c+1,j} \leq q_{c,j-1}$. Therefore, if $q_{c,j} < q_{c+1,j}$ then we derive a contradiction.

• **Case 2b.** Identical to Case 2a.

• **Case 2c.** Suppose $q_{c,j} = q_{c+1,j}, l_{c,j} > l_{c+1,j}$ and $q_{c,j-1} > q_{c+1,j-1}$. As $q_{c,j-1} \neq q_{c+1,j-1}$ then $q_{c+1,j-1} = q_{c+1,j}$ (line 12). We also know $q_{c,j-1} \leq q_{c,j} \leq q_{c+1,j} \leq q_{c,j-1}$ from Case 1a. Putting everything together, we get $q_{c,j-1} \leq q_{c,j} \leq q_{c+1,j-1} < q_{c,j-1}$. This leads to a contradiction.

• **Case 2d.** Suppose $q_{c,j} = q_{c+1,j}, l_{c,j} < l_{c+1,j}$ and $q_{c,j-1} = q_{c+1,j-1}, l_{c,j-1} \geq l_{c+1,j-1}$. As we know from

Case 1a $q_{c+1,j-1} \leq q_{c+1,j} \leq q_{c,j-1}$, $q_{c,j-1} \leq q_{c,j} \leq q_{c-1,j-1}$ and $q_{c,j-1} \leq q_{c,j} \leq q_{c+1,j} \leq q_{c,j-1}$. Hence, $q_{c+1,j-1} = q_{c+1,j} = q_{c,j-1} = q_{c,j}$.

Consider two subcases. Suppose $q_{c,j-1} < q_{c-1,j-1}$. Then $l_{c,j} = \infty$ (line 12). Hence, our assumption $l_{c,j} < l_{c+1,j}$ is false.

Suppose $q_{c,j-1} = q_{c-1,j-1}$. If $l_{c-1,j-1} = len$ then $l_{c,j} = \infty$ (line 12). Hence, our assumption $l_{c,j} < l_{c+1,j}$ is false. Therefore, $l_{c-1,j-1} \in [1, len)$ and $l_{c,j-1} = l_{c-1,j-1} + 1$. By induction hypothesis as $q_{c+1,j-1} = q_{c,j-1} = q_{c-1,j-1}$ then $l_{c+1,j-1} \leq l_{c,j-1} \leq l_{c-1,j-1}$. Hence, $l_{c,j-1} \in [1, l_{c-1,j-1}] \subseteq [1, len)$. Therefore, $l_{c+1,j} = l_{c,j-1} + 1 \leq l_{c-1,j-1} + 1 = l_{c,j-1}$. This contradicts our assumption $l_{c,j} < l_{c+1,j}$.

Case 3. Consider the case $x_j \in N_k$. This case follows immediately from Algorithm 3, line 14, and the induction hypothesis. \square

F Proof of Lemma 6

Proof of correctness. We prove by induction on the length of the sequence. Given $f_{c,j}$ we can reconstruct a corresponding set of sequences $S_{c,j}$ by traversing the table backward.

The base case is trivial as $x_1 \in P_k$, $f_{0,0} = \{1, 1\}$ and $f_{c,0} = \{\infty, \infty\}$. Suppose the statement holds for $j - 1$ variables.

Case 1. Consider the case $x_j \in P_k$. Note, that the cost can not be increased on seeing $x_j \in P_k$ as cost only depends on covered undetermined variables. By the induction hypothesis, $S_{c,j-1}$ satisfies conditions 1–4. The only way to obtain $S_{c,j}$ from $S_{c',j-1}$, $c' \in [0, z_c^U]$, is to extend $last(S_{c',j-1})$ to cover x_j or start a new sequence if $|last(S_{c',j-1})| = len$. If $S_{c',j-1}$ does not exist then $S_{c,j}$ does not exist. The algorithm performs this extension (lines 9 and 10). Hence, $S_{c,j}$ satisfies conditions 1–4.

Case 2. Consider the case $x_j \in U_k$. In this case, there exist two options to obtain $S_{c,j}$ from from $S_{c',j-1}$, $c' \in [0, z_c^U]$.

The first option is to cover x_j . Hence, we need to extend $last(S_{c-1,j-1})$. Note that we should not start a new sequence if $last(S_{c-1,j-1}) = len$ as it is never optimal to start a sequence on seeing a neutral variable.

The second option is not to cover x_j . Hence, we need to interrupt $last(S_{c,j-1})$.

By Lemma 5 we know that $f_{c,j-1} \leq f_{c-1,j-1}$, $0 < c \leq C$. By the induction hypothesis, $S_{c,j-1}$ and $S_{c-1,j-1}$ satisfy conditions 1–4. Hence, $S_{c,j-1} \leq S_{c-1,j-1}$.

Consider two cases. Suppose $|S_{c,j-1}| < |S_{c-1,j-1}|$. In this case, it is optimal to interrupt $last(S_{c,j-1})$.

Suppose $|S_{c,j-1}| = |S_{c-1,j-1}|$ and $|last(S_{c,j-1})| \leq |last(S_{c-1,j-1})|$. If $|last(S_{c-1,j-1})| < len$ then it is optimal to extend $last(S_{c-1,j-1})$. If $|last(S_{c-1,j-1})| = len$ then it is optimal to interrupt $last(S_{c,j-1})$, otherwise we would have to start a new sequence to cover an undetermined variable x_j , which is never optimal. If $S_{c,j-1}$ and $S_{c-1,j-1}$ do not exist then $S_{c,j}$ does not exist. If $S_{c,j-1}$ does not exist then case analysis is similar to the analysis above.

This case-based analysis is exactly what Algorithm 3 does in lines 12 and 12. Hence, $S_{c,j}$ satisfies conditions 1–4.

Case 3. Consider the case $x_j \in N_k$. Note that the cost can not be increased on seeing $x_j \in N_k$ as cost only depends on covered undetermined variables. By the induction hypothesis, $S_{c,j-1}$ satisfies conditions 1–4. The only way to obtain $S_{c,j}$ from $S_{c',j-1}$, $c' \in [0, z_c^U]$, is to interrupt $last(S_{c',j-1})$. If $S_{c',j-1}$ does not exist then $S_{c,j}$ does not exist. The algorithm performs this extension in line 14. Hence, $S_{c,j}$ satisfies conditions 1–4. \square

Proof of complexity. The time complexity of the algorithm is $O(n \max(z_c)) = O(n^2)$ as we have $O(n \max(z_c))$ elements in the table and we only need to inspect a constant number of elements to compute $f(c, j)$. \square

G Proof of Lemma 8

First, we present explicitly the algorithm for detecting disentanglement.

Algorithm 4: WEIGHTEDSPRINGYFOCUS(x_0, \dots, x_{n-1})

```

1 for  $c \in -1..z_c^U$  do
2   for  $j \in -1..n - 1$  do
3      $f_{c,j} \leftarrow \{\infty, \infty, \infty\}$ ;
4    $f_{0,-1} \leftarrow \{0, 0, 0\}$ ;
5   for  $j \in 0..n - 1$  do
6     for  $c \in 0..j$  do
7       if  $x_j \in P_k$  then /* penalizing */
8         if  $(l_{c,j-1} \in [1, len]) \vee (q_{c,j-1} = \infty)$  then
9            $f_{c,j} \leftarrow \{q_{c,j-1}, l_{c,j-1} + 1, h_{c,j-1}\}$ ;
10        else
11           $f_{c,j} \leftarrow \{q_{c,j-1} + 1, 1, 0\}$ ;
12        if  $x_j \in U_k$  then /* undetermined */
13          if  $(l_{c-1,j-1} \in [1, len] \wedge q_{c-1,j-1} =$ 
14             $q_{c,j-1}) \vee (q_{c,j-1} = \infty)$  then
15             $f_{c,j} \leftarrow \{q_{c-1,j-1}, l_{c-1,j-1} + 1, h_{c-1,j-1}\}$ 
16          else
17             $f_{c,j} \leftarrow \{q_{c,j-1}, \infty, \infty\}$ 
18          if  $x_j \in N_k$  then /* neutral */
19            if  $(l_{c-1,j-1} \in [1, len] \wedge h_{c-1,j-1} \in$ 
20               $[1, h] \wedge q_{c-1,j-1} = q_{c,j-1}) \vee (q_{c,j-1} = \infty)$  then
21               $f_{c,j} \leftarrow \{q_{c-1,j-1}, l_{c-1,j-1} + 1, h_{c-1,j-1} + 1\}$ 
22            else
23               $f_{c,j} \leftarrow \{q_{c,j-1}, \infty, \infty\}$ 
24          return  $f$ ;
```

The algorithm is based on a dynamic program. Each cell in the dynamic programming table $f_{c,j}$, $c \in [0, z_c^U]$, $j \in \{0, 1, \dots, n - 1\}$, where $z_c^U = \max(z_c) - |P_k|$, is a triple of values $q_{c,j}$, $l_{c,j}$ and $h_{c,j}$, $f_{c,j} = \{q_{c,j}, l_{c,j}, h_{c,j}\}$, stores information about $S_{c,j}$. The new parameter $h_{c,j}$ stores the number of neutral variables covered by $last(S_{c,j})$. Algorithm 4 shows pseudocode of the algorithm.

The intuition behind the algorithm is as follows. We emphasize again that by cost of a cover we mean the number of covered undetermined and neutral variables.

- If $x_j \in P_k$ then we do **not** increase the cost of $S_{c,j}$ compared to $S_{c,j-1}$ as the cost only depends on $x_j \in U_k$. Hence, the best move for us is to extend $last(S_{c,j-1})$ or we start a new sequence if it is possible. This is encoded in lines 9 and 11 of the algorithm.

- If $x_j \in U_k$ then we have two options. We can obtain $S_{c,j}$ from $S_{c-1,j-1}$ by increasing $cst(S_{c-1,j-1})$ by one. This means that x_i will be covered by $last(S_{c,j})$. Note that this does not increase the number of covered neutral variables by $last(S_{c,j})$ as we can always set $x_j = v, v > k$. Alternatively, from $S_{c,j-1}$ by interrupting $last(S_{c,j-1})$ if necessary. This is encoded in lines 14 and 16 of the algorithm.
- If $x_j \in N_k$ then we have two options. We can obtain $S_{c,j}$ from $S_{c-1,j-1}$ by increasing $cst(S_{c-1,j-1})$ by one and increasing the number of covered neutral variables by $last(S_{c,j-1})$. Alternatively, from $S_{c,j-1}$ by interrupting $last(S_{c,j-1})$ (lines 19–21).

The proof of correctness mimics the corresponding proof for the WEIGHTEDFOCUS constraint. We can now prove Lemma 8.

Proof. The main idea is identical to the proof of the WEIGHTEDFOCUS constraint. We only highlight the differences between the WEIGHTEDFOCUS constraint and the WEIGHTEDSPRINGYFOCUS constraint.

Consider a variable-value pair $x_i = v, v > k$. The only difference is in the fourth option. We denote $h(s_{i,j})$ the number of neutral variables covered by $s_{i,j}$. Similarly, $h(S) = \sum_{s_{i,j} \in S} h(s_{i,j})$.

- The fourth and the cheapest option is to glue $last(S_{c_1,i-1}), x_v$ and $last(S_{c_2,n-i-2})$ to a single sequence if $|last(S_{c_1,i-1})| + |last(S_{c_2,n-i-2})| < len$ and $h(last(S_{c_1,i-1})) + h(last(S_{c_2,n-i-2})) \leq h$. Hence, $S'_{c_1,i-1} = S_{c_1,i-1} \setminus last(S_{c_1,i-1})$, $S'_{c_2,n-i-2} = S_{c_2,n-i-2} \setminus last(S_{c_2,n-i-2})$ and s' is a concatenation of $last(S_{c_1,i-1}), x = v$ and $last(S_{c_2,n-i-2})$. Then the union $S = S'_{c_1,i-1} \cup S'_{c_2,n-i-2} \cup \{s'\}$ forms a cover: $cst(S) = c_1 + c_2 + 1, |S| = |S_{c_1,i-1}| + |S_{c_2,n-i-2}| - 1$ and $h(S) = h(last(S_{c_1,i-1})) + h(last(S_{c_2,n-i-2}))$.

The rest of the proof is analogous to WEIGHTEDFOCUS.

Consider a variable-value pair $x_i = v, v \leq k$. The main difference is that we have the second option to build a support. Namely, we glue $S_{c_1,i-1}, x_i$ and $S_{c_2,n-i-2}$. Hence, if $c_1 + c_2 + 1 \leq z_c^U, |last(S_{c_1,i-1})| + |last(S_{c_2,n-i-2})| < len$ and $h(last(S_{c_1,i-1})) + h(last(S_{c_2,n-i-2})) < h$ then we can build a support for $x_i = v$. The rest of the proof is analogous to WEIGHTEDFOCUS. \square